ON THE GENERALIZED APPROACHES IN STATISTICAL PROCESS CONTROL

BY

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To my dear sister (late)

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Rashid Mehmood

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LIST OF ABBREVIATIONS

PTS	:	Probability to signals
AADP	:	Average absolute difference of probability to signals
DMD	:	Diffuse mean disturbances
DAVD	:	Diffuse asymmetric variance disturbances
BT	:	Bivariate t
BLN	:	Bivariate lognormal
BN	:	Bivariate normal
LMD	:	Localized mean disturbances
LVD	:	Localized variance disturbances
SRS	:	Simple random sampling
RSS	:	Ranked set sampling
MRSS	:	Median ranked set sampling
ERSS	:	Extreme ranked set sampling
PRSS	:	Percentile ranked set sampling
DRSS	:	Double ranked set sampling
DMRSS	:	Double median ranked set sampling
DERSS	:	Double extreme ranked set sampling

MDRSS	:	Median double ranked set sampling
EDRSS	:	Extreme double ranked set sampling
SC	:	Skewness correction
R	:	Range
MAD	:	Median absolute deviation

ABSTRACT

Full Name : [RASHID MEHMOOD]

Thesis Title : [ON THE GENERALIZED APPROACHES IN STATISTICAL PROCESS CONTROL]

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In this thesis we developed generalized structures of Shewhart control charts for the monitoring of location and dispersion parameters. In order to develop these structures we used robust dispersion estimators, auxiliary information based on location and dispersion estimators, and an efficient use of Cornish fisher expansion in the form of skewness correction method. For performance evaluation we considered false alarm rate and probability to signals as performance measures. The outcomes of the thesis showed that the new design structures of Shewhart control charts perform outstandingly compared to the counterparts. Also, new designed schemes can be treated as generalized forms of some existing schemes. Moreover, practical applications and numerical illustrations are also included to verify the study for practical purposes.

ملخص الرسالة

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في هذه الرسالة قمنا بتطوير نماذج عامة من لوحات شيوارت للتحكم لرصد معالم الموقع والتشتت. من أجل تطوير هذه النماذج استخدمنا مقدرات تشتت قليلة التأثر، معلومات مساعدة مبنيّة على معالم الموقع والتشتت، واستخدام فعال لتمدد كورنيش فيشر في شكل طريقة تصحيح الانحراف. أما بالنسبة لتقييم الأداء فقد استخدمنا معدل الانذار الكاذب ونسبة الاحتمال إلى الإشارات. أظهرت نتائج الرسالة أن أداء النماذج الجديدة للوحات شيوارت للتحكم يفوق أداء لوحات التحكم النظيرة. كذلك أظهرنا أن الخطط الجديدة يمكن أن تعامل على أنها أشكال عامة من بعض الخطط الموجودة. أضف إلى ذلك، أضممنا تطبيقات عملية وأمثلة رقمية توضيحية للتحقق من الدراسة في الأغراض العمليّة.

CHAPTER 1

INTRODUCTION

This chapter is planned to provide a brief introduction of statistical process control (SPC) and more specifically about quality control chart. Also, we include an outline of the thesis.

1.1 Statistical process control

Output measures follow probability distribution. For any distribution there is one or more than one parameter (also termed as process parameter). The parameter can be location, dispersion, and shape. The stability of these parameters is very important as process associated with certain variations. These variations are categorized as common cause and/or special cause. Common cause variation is also termed as chance cause variation, non-assignable cause variation, noise, or natural variation. Common cause variation is usually small in magnitude and can occur due to many reasons such as poor design, poor maintenance of machines, poor working conditions (e.g. lighting, noise, and temperature), substandard raw material, measurement error etc. On the other hand, special cause of variations also known as assignable cause, signal, and un-natural variation. This type of variation is generally large in magnitude and appeared as a consequence of poor adjustment of equipment, computer crash, high healthcare demand, poor batch of raw material, sudden breakdown of circuits etc.

The following variations in the process parameter (s) can be analyzed through statistical process control (SPC). Statistical process control (SPC) is a technique of quality control which includes variety of tools such as control charts, check sheets, pareto diagram, and histogram. Nowadays, among these tools the most popular one is the quality control chart.

1.2 Control chart

Control chart is a powerful tool to monitor the variations in process parameters (location and dispersion), which was initiated by Shewhart in 1931. The most frequent Shewhart-type control charts are mean ($\bar{X}$) for location parameter, standard deviation (S), and variance (S^2) for process dispersion. The design structure of a control chart depends on a centerline (CL), upper control limit (UCL) and lower control limits (LCL). A process cannot be declared out-of-control until any plotted statistic (such as $\bar{X}$, S and S^2) breach the control limits (UCL and LCL). Quality control charts have wide application in several disciplines such as environmental sciences (see Gove et al., 2013), industrial process (see Riaz et al., 2011), agriculture (see Mehmood et al., 2013b), analytical measurement (see Abbasi, 2010), medical sciences (see Weiß & Atzmüller , 2010).

1.3 Outline of the thesis

In Chapter 2 a new design structure of dispersion control charts is developed by merging the variety of robust estimators under different ranked set strategies and runs rules. The new structures are evaluated in term of power. The results indicated that ranked set strategies based dispersion control charts triggered more signals than the usual existing control chart. Among different sampling strategies, DMRSS outperforms the others followed by MDRSS, MRSS and PRSS in the said order. The performance order of different estimators under both single and double ranked set strategies shows that S and

GINI stay on superior end while *R*, *MAD* and Q_n charting structures offer relatively lower performance under different ranked set strategies with varying runs rules schemes. An article entitled on "On the application of different ranked set sampling schemes" has been published in *Quality Engineering* (see Mehmood et al., 2014).

In chapter 3 we incorporate a variety of runs rules with the design structure of usual $\bar{X}$, R and S^2 control chart by using skewness correction method. We have investigated the performance of the proposals in terms of false alarm rate and signaling probability. The outcomes of the proposal serve the dual purpose in terms of robustness and efficiency. The developed schemes in the chapter are generalized forms of various studies (e.g. Shewhart (1931), Chan & Cui (2003) and Riaz et al. (2011)). One article based on this chapter has been accepted for publication in *Quality and Reliability Engineering International*.

Chapter 4 offers an extended design structure of existing dual auxiliary information based location control charts by integrating three bivariate distributions, double ranked set strategies and runs rules. Additionally, in the chapter we proposed a design structure for unknown skewed distributions using the skewness correction method. The developed structure depends upon the skewness of the study variable, amount of correlation between study variable and auxiliary variable, and sample size. The concluding remarks of this chapter revealed that extreme ranked set strategies based control charts are more robust than ranked set strategies based control charts. It is important to mention that location control charts proposed by Abbasi & Riaz (2015) are all special cases of our proposed methodology.

In chapter 5 we follow Abbasi & Riaz (2015) and introduce a new design structure of variance control charts. The design structure has ability to utilize the auxiliary information for ranking the units of interest as well as for estimation of parameters (dual purpose). The study conducted by Riaz et al. (2014) is treated as the special case of our study.

CHAPTER 2

IMPROVED DISPERSION CONTROL CHARTING UNDER DIFFERENT SAMPLING STRATEGIES

In this chapter we propose dispersion control charting structures based on five different estimators, under a variety of sampling strategies and runs rules schemes. We use power curves to compare the control charts. The dispersion charts under varying runs rules schemes with different sampling strategies improve significantly the detection ability of these charts. In particular, the sample standard deviation and the sample *Gini's* mean differences based structures with median ranked set sampling perform well under both single and double ranked set strategies with varying runs rules schemes. We include a real life example to illustrate the procedural details and to highlight their practical significance.

2.1 Introduction

In many practical circumstances we have some extra information about the study variable that will be helpful to measure the variable of interest. This information may be used to make sampling mechanisms more effective. McIntyre (1952) introduced the idea of ranked set sampling (RSS) as a substitute of simple random sampling (SRS).

The sampling methodology used in deriving the usual control charting structures is SRS. Control charts for monitoring the process mean level based on RSS are described in Muttlak & Al-Sabah (2003) among others. Mehmood et al. (2013b) studied control charts for monitoring the process mean level under a wide variety of sampling plans. The sampling schemes were based on SRS, RSS, median ranked set sampling (MRSS), extreme ranked set sampling (ERSS), percentile ranked set sampling (PRSS), double ranked set sampling (DRSS), double median ranked set sampling (DMRSS), median double ranked set sampling (MDRSS), double percentile ranked set sampling (DPRSS) and percentile double ranked set sampling (PDRSS). Furthermore, they carry out extensive comparisons among these strategies under different runs rules schemes (cf. Riaz et al., 2011). The recommendation of Mehmood et al. (2013b) was to use median ranked set sampling strategies with the implementation of varying runs rules schemes.

Schoonhoven et al. (2011) and Abbasi & Miller (2012) carried out detailed studies for the dispersion parameter and conducted the proper choice of a control chart for dispersion by investigating a wide range of estimators under SRS. The dispersion control charts using ranked set strategies have not been explored extensively. One of the few examples is the paper of Abujiya & Muttlak (2007), where extreme variations of first and second stage RSS are used to improve the Shewhart range (R) control chart.

In this chapter we choose some selective estimators based on the studies of Schoonhoven et al. (2011) and Abbasi & Miller (2012), and explore the properties of their design structures under different sampling strategies by implementing a variety of runs rules schemes. The sampling strategies covered in the current chapter include SRS, MRSS, PRSS, DMRSS, and MDRSS (this selection of sampling strategies is based on Mehmood et al. (2013b). The selection of the runs rules schemes is based on the results of Riaz et al. (2011). Furthermore, we carry out extensive comparisons among these dispersion design structures and sampling strategies under different runs rules schemes.

This chapter is organized as follows: In section 2.2 we provide the conceptual framework of the different sampling strategies; Section 2.3 provides different dispersion estimators under varying sampling schemes; In Section 2.4 we work out the design structures of the proposed runs rules based control charts under different sampling strategies for monitoring the dispersion parameter; In Section 2.5 we evaluate and compare the proposed control charts using power curves as performance measure; In Section 2.6 we provide a numerical example for illustrative purposes; Finally, Section 2.7 provides the summary and main findings of the study.

2.2 Sampling strategies

Ranked set methods are mainly classified into two categories, namely single and double ranked set sampling strategies. We also cover the cases of both perfect and imperfect ranking for selecting the ranked set data.

The detailed procedure of collecting the RSS sample (single and double) may be found in Mehmood et al. (2013b). A brief description is: In order to gather a ranked set sample of size n, draw n random samples using SRS each of size n units. After that, assign ranks to each unit of the samples. For the said purposes we select the smallest unit from the first sample, the second smallest unit from the second sample and so on until the largest unit is chosen for the *nth* sample. This mechanism results into a ranked set sample of size n. By repeating the prescribed procedure r times we attain nr ranked set samples. In order to get MRSS, we select the median order statistic for each sample. The PRSS is similar to MRSS in the sense that the median order statistic is replaced by a particular percentile (e.g. the 30th percentile) order statistic according to our interest. In ERSS we select the smallest observation from the first half sets and the largest observation from the second half sets. These are called single ranked set sampling methods.

We may easily extend these definitions to the double ranked set sampling techniques. If we collect n ranked set samples each of size n at the first stage, followed by implementing the ranked set procedure on the collected ranked set sample again, then the resulting sample is called the double ranked set sample. An RSS at the first stage followed by RSS again at the second stage gives us DRSS. A first stage RSS followed by MRSS at the second stage results into MDRSS while a second stage PRSS ends with DPRSS. For more details one may see Mehmood et al. (2013b).

All above mentioned sampling mechanisms are called perfect ranked set sampling (based on the visual inspection). Visual inspection is not always possible which leads to imperfect ranked set sampling and hence we use another variable, called concomitant variable (Y) that helps in ranking the study variable (X). We will denote the correlation between X and Y by ρ. The procedure of selecting the sample starts with generating bivariate random samples (X_i, Y_i) each of size n. Rank the variable of interest (X) with respect to concomitant variable (*Y*). By ranking *X* with respect to *Y*, the steps for the rest of the selection procedure are the same as that of the perfect ranked set sampling. The resulting sample is called the imperfect ranked set sampling (IRSS). By following this procedure we can implement this idea for all sampling strategies given above for single and double sampling strategies.

It may be noted that the varying values of ρ lead to different sampling mechanisms: $\rho = 0$ results into SRS, $\rho = 1$ leads to perfect ranking and $0 < \rho < 1$ gives imperfect ranking (cf. Abujiya & Muttlak, 2004).

2.3 Dispersion estimators

Based on the recommendation of Mehmood et al. (2013b), we mainly consider some selective sampling strategies, namely MRSS, PRSS, MDRSS and DMRSS, along with SRS. Let T denote one of the different sampling strategies MRSS, PRSS, MDRSS and DMRSS under both single and double ranked set sampling strategies.

Assume that μ_X and σ_X are the location and dispersion parameters of the quality characteristic of interest *X*. We study the following dispersion estimators: standard deviation, range, *Gini's* mean differences, mean absolute deviation of the median and a robust alternative denoted by Q_n, (cf. Schoonhoven et al., 2011; and Abbasi & Miller, 2012).

2.3.1 Dispersion Estimators Based on Single Ranked Set Strategies

Let $X_{(i:m)j}$ be the i^{th} median of the quality characteristic of interest *X* for the i^{th} SRS set of size *n* in the j^{th} cycle, then the dispersion estimators under MRSS are defined as:

$$S_{M,j} = \sqrt{\sum_{i=1}^{n}\left(X_{(i:m)j} - \bar{X}_{M,j}\right)^2 \Big/ (n-1)} \quad \text{, where } \bar{X}_{M,j} = \sum_{i=1}^{n} X_{(i:m)j} / n$$

$$R_{M,j} = max(X_{(i:m)j}) - min(X_{(i:m)j})$$

$$GINI_{M,j} = \frac{\sqrt{\pi}}{2} \sum_{\substack{l=1 \\ l \neq q}}^{n} \sum_{q=1}^{n} \left| X_{(l:m)j} - X_{(q:m)j} \right| \Big/ \binom{n}{2}$$

$$MAD_{M,j} = 1.4826 med_i \left| X_{(i:m)j} - \tilde{X}_{M,j} \right|$$

$Q_{n_{M,j}} = 2.2219 \left\{ \left| X_{(l:m)j} - X_{(q:m)j} \right| ; l < q \right\}_{(k)}$; where $k = \binom{[n/2]+1}{2}$ refers to the k^{th} order statistic of the $\frac{n!}{2!(n-2)!}$ inter-point distances.

On the same lines one can define the corresponding five dispersion estimators (given above) under PRSS.

2.3.2 Dispersion Estimators Based on Double Ranked Set Strategies

Let $G_{(i:m)j}$ be the i^{th} median of the quality characteristic of interest *X* for the i^{th} RSS set of size *n* in the j^{th} cycle, then the dispersion estimators under MDRSS are defined as:

$$S_{MD,j} = \sqrt{\sum_{i=1}^{n}\left(G_{(i:m)j} - \bar{X}_{MD,j}\right)^2 \Big/ (n-1)} \quad \text{, where } \bar{X}_{MD,j} = \sum_{i=1}^{n} G_{(i:m)j} / n$$

$$R_{MD,j} = max(G_{(i:m)j}) - min(G_{(i:m)j})$$

$$GINI_{MD,j} = \frac{\sqrt{\pi}}{2} \sum_{\substack{l=1 \\ l \neq q}}^{n} \sum_{q=1}^{n} \left| G_{(l:m)j} - G_{(q:m)j} \right| \Big/ \binom{n}{2}$$

$$MAD_{MD,j} = 1.4826 med_i \left| G_{(i:m)j} - \tilde{X}_{MD,j} \right|$$

$Q_{n_{MD,j}} = 2.2219 \left\{ \left| G_{(l:m)j} - G_{(q:m)j} \right| ; l < q \right\}_{(k)}$; where (k) refers to the k^{th} order statistic of the $\frac{n!}{2!(n-2)!}$ inter-point distances for a MDRSS of size *n*.

Let $D_{(i:m)j}$ be the i^{th} median of the quality characteristic of interest X for the i^{th} MRSS set of size n in the j^{th} cycle, then the corresponding five dispersion estimators under DMRSS can be defined in the same way.

2.3.3 Dispersion Estimators Based on Imperfect Ranked Set Strategies

For the case of imperfect ranking the above mentioned dispersion estimators may be easily redefined for the different ranked set sampling strategies. Let $(X_{[i,n],j}, Y_{(i:n),j})$ denote the pair of i^{th} smallest value of Y associated with the corresponding value of X obtained from the i^{th} set in the j^{th} cycle, then the dispersion estimators under imperfect median ranked set sampling (IMRSS) are defined as:

$$S_{IM,j} = \sqrt{\sum_{i=1}^{n}\left(X_{[i:m],j} - \bar{X}_{IM,j}\right)^2 \Big/ (n-1)}$$

$$R_{IM,j} = max(X_{[i:m],j}) - min(X_{[i:m],j})$$

$$GINI_{IM,j} = \frac{\sqrt{\pi}}{2}\sum_{\substack{l=1\\ l\neq q}}^{n}\sum_{q=1}^{n}\left|X_{[l:m],j} - X_{[q:m],j}\right| \Big/ \binom{n}{2}$$

$$MAD_{IM,j} = 1.4826 med_i \left|X_{[i:m],j} - \tilde{X}_{IM,j}\right|$$

$$Q_{n_{IM,j}} = 2.2219\left\{\left|X_{[l:m],j} - X_{[q:m],j}\right|; l < q\right\}_{(k)}$$

The idea of imperfect ranking may easily be applied to the other single and double ranked set sampling and the corresponding dispersion estimators may be defined on the similar lines, as defined above for the case of IMRSS.

2.4 Proposed control charting structures

In this section we propose a generalized set of runs rules based on control charting structures for monitoring the dispersion parameter of the ongoing process. The sampling strategies for collecting the sample may be any one from section 2.2 and the dispersion estimator may be chosen as given in section 2.3. For the ease of practitioners and readers, we denote the selected dispersion estimator (*S, R, GINI, MAD* and Q_n) by using capital "*E*" and the chosen sampling strategies (MRSS, PRSS, DMRSS, MDRSS) are denoted by subscript "*T*". Hence, the dispersion estimators are denoted by 'E_T' (e.g. $S_T, R_T, GINI_T, Q_{n_T}, MAD_T$). We use the runs rules as proposed by Riaz et al. (2011) who implemented these runs rules for the Shewhart control chart under SRS:

A process is declared as out-of-control if at least *k-m* of *k* consecutive points (where $0 \le m \le k-1$) of the ongoing process falls outside the control limits (H_L, H_U) of the sampling distribution of the control charting statistic E_T. For more convenience, we can write it in the form: $k-m/k$. (cf. Riaz et al., 2011).

Here the signaling limits H_L and H_U for the two sided control structures of the dispersion charts are given by:

$$H_L = A_{(T,E_T,n,\rho,m,k,p/2)}\,\sigma_X \quad , \quad H_U = A_{(T,E_T,n,\rho,m,k,(1-p/2))}\,\sigma_X \qquad \textbf{(2.1)}$$

The coefficient '*A*' is a function of the sampling strategy (*T*), dispersion estimator (*E*), sample size (*n*), and correlation between *X* and *Y* (ρ). Here k represents the total consecutive points to be considered, k-m are the decision observations used in a given rule and p is the probability of a single point falling outside the respective signaling limits depending upon *k-m* and *k.*

The probability of a single point falling outside the respective signaling limits p, mentioned in (2.1) for a given runs rules scheme depends on m, k and α (the pre-specified false alarm rate) and may be obtained by solving the following equation:

$$\alpha = \sum_{k-m\leq k} \frac{k!}{(k-m)!m!} p^{(k-m)}(1-p)^{m} \ , where\ 0 \leq m \leq k-1 \qquad \textbf{(2.2)}$$

The values of p for different rules investigated in this study are listed in Table 2.1 for $\alpha = 0.0027$. Similar values may be obtained for any choice of α by solving (2.2).

Table 2.1: Values of p for Different Sensitizing Rules

Rule	**1/1**	**2/2**	**1/2**	**3/3**	**2/3**	**1/3**	**4/4**	**3/4**
p	0.0027	0.0014	0.1392	0.1392	0.0303	0.0009	0.008	0.0898
Rule	**2/4**	**5/5**	**4/5**	**3/5**	**6/6**	**5/6**	**4/6**	**7/7**
p	0.0215	0.1577	0.0669	0.0669	0.3732	0.2231	0.1219	0.4296
Rule	**6/7**	**5/7**	**8/8**	**7/8**	**6/8**	**9/9**	**8/9**	**7/9**
p	0.2826	0.4774	0.3554	0.3554	0.2304	0.5183	0.3821	0.2788

For the case of unknown parameters (2.1) may be written with the estimated quantities as:

$$H_L = A_{(T,E_T,n,\rho,m,k,p/2)} \frac{\bar{E}_T}{d_{2(E,T)}} \quad , \quad H_U = A_{(T,E_T,n,\rho,m,k,(1-p/2))} \frac{\bar{E}_T}{d_{2(E,T)}} \qquad \textbf{(2.3)}$$

where $\bar{E}_T$ represents the mean of the corresponding dispersion statistic (cf. Section 2.3) for a given sampling strategy T (cf. Section 2.2), $d_{2(E,T)}$ represents the unbiasing constant for a given dispersion statistic E and sampling strategy T (i.e. E_T). For some selective choices the values of $d_{2(E,T)}$ are provided in Table 2.2. The control structure given in (2.1) and the corresponding estimated structure in (2.3) represents different dispersion charting structures, namely *S*, *Range*, *GINI*, *MAD*, and Q_n control charts (depending on the choice of T and E_T). These structures are capable to take care of certain issues (like biasedness and non-monotonicity) and provide simpler design structures by avoiding simultaneous application of many runs rules (cf. Does & Schriever, 1992; and Riaz et al., 2011).

Table 2.2. Unbiasing Constants for Different Ranked Set Based Charts

T	ρ / n	0.25			0.50			0.75			1		
		3	5	7	3	5	7	3	5	7	3	5	7
S	**MRSS**	0.88	0.93	0.92	0.83	0.85	0.84	0.74	0.73	0.72	0.59	0.51	0.5
	DMRSS	0.87	0.91	0.95	0.8	0.83	0.84	0.66	0.64	0.65	0.4	0.27	0.2
	MDRSS	0.87	0.93	0.93	0.81	0.85	0.85	0.69	0.68	0.67	0.48	0.35	0.28
	PRSS	0.88	0.92	0.92	0.83	0.85	0.85	0.74	0.74	0.72	0.59	0.53	0.52
R	**MRSS**	1.69	2.3	2.27	1.58	2.1	2.1	1.41	1.81	2.04	1.12	1.26	1.25
	DMRSS	1.66	2.25	2.68	1.52	2.05	2.37	1.26	1.6	1.84	0.77	0.67	0.57
	MDRSS	1.66	2.29	2.63	1.54	2.09	2.39	1.32	1.67	1.89	0.91	0.88	0.81
	PRSS	1.69	2.27	2.27	1.58	2.1	2.1	1.41	1.82	2.04	1.12	1.3	1.29
GINI	**MRSS**	1.13	1.11	1.1	1.05	1.02	1.01	0.94	0.88	0.84	0.75	0.61	0.6
	DMRSS	1.11	1.09	1.12	1.01	0.99	0.99	0.84	0.77	0.77	0.51	0.32	0.24
	MDRSS	1.11	1.11	1.1	1.03	1.01	1	0.88	0.81	0.79	0.6	0.42	0.33
	PRSS	1.13	1.1	1.1	1.05	1.02	1.02	0.94	0.88	0.85	0.75	0.64	0.62
MAD	**MRSS**	0.72	0.62	0.61	0.67	0.58	0.56	0.6	0.49	0.49	0.48	0.33	0.33
	DMRSS	0.7	0.61	0.66	0.64	0.56	0.58	0.53	0.43	0.44	0.32	0.18	0.14
	MDRSS	0.7	0.62	0.64	0.65	0.56	0.59	0.56	0.45	0.46	0.38	0.24	0.19
	PRSS	0.72	0.61	0.61	0.67	0.56	0.56	0.6	0.49	0.5	0.48	0.36	0.35
Q_n	**MRSS**	1.03	1.19	1.17	0.94	1.09	1.07	0.86	0.93	0.86	0.69	0.63	0.64
	DMRSS	1.01	1.15	1.17	0.89	1.03	1.02	0.75	0.82	0.79	0.46	0.34	0.25
	MDRSS	0.99	1.17	1.13	0.89	1.05	1.05	0.8	0.84	0.81	0.54	0.45	0.35
	PRSS	1.03	1.16	1.17	0.94	1.08	1.08	0.86	0.92	0.87	0.69	0.68	0.66

The coefficients A are obtained through Monte Carlo simulation following Riaz et al. (2011) and Mehmood et al. (2013b). The procedure to carry out the simulations may be described as follows: For a given choice of $n, T\ and\ \rho$ we generate random samples from the standard bivariate distribution (without loss of generality) and calculate the charting statistic (E_T) for each sample. We repeat this mechanism 10^5 to 10^6 times using the combinations of the aforementioned quantities to obtain values of E_T. For a pre-specified false alarm rate α we pick the respective quantiles either on one side of or both sides of the tails. In this way the control limits multiplier A are obtained for the given information of the aforementioned quantities and varying values of *m*, *k* and *p*. The resulting outcomes are tabulated in the form of Tables 2.3-2.7 for a given estimator (*S*, *R*, *GINI*, *MAD* and Q_n) under MRSS at $\alpha = 0.0027$ (upper sided).

Table 2.3. Control Limits Multipliers of *S* control chart based on MRSS for different runs rules at $\alpha = 0.0027$

ρ	*n*	Rules					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0	**3**	2.43	1.87	1.96	0.81	0.98	1.13
	5	2.01	1.63	1.69	0.9	1.02	1.13
	7	1.83	1.53	1.58	0.93	1.03	1.12
0.25	**3**	2.4	1.84	1.94	0.8	0.96	1.11
	5	1.97	1.6	1.66	0.88	1	1.1
	7	1.78	1.49	1.53	0.91	1.01	1.09
0.50	**3**	2.26	1.74	1.82	0.75	0.91	1.05
	5	1.83	1.48	1.54	0.82	0.93	1.02
	7	1.64	1.36	1.41	0.83	0.92	1
0.75	**3**	2.02	1.55	1.63	0.67	0.81	0.94
	5	1.56	1.26	1.31	0.7	0.79	0.87
	7	1.37	1.14	1.17	0.69	0.77	0.83
1	**3**	1.64	1.25	1.31	0.54	0.66	0.76
	5	1.08	0.88	0.91	0.48	0.55	0.6
	7	0.85	0.7	0.73	0.43	0.47	0.52

Table 2.4. Control Limits Multipliers of *R* control chart based on MRSS for different runs rules at $\alpha = 0.0027$

ρ	n	Rules					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0	**3**	4.68	3.59	3.76	1.54	1.87	2.15
	5	5.12	4.1	4.26	2.22	2.53	2.79
	7	5.33	4.38	4.53	2.59	2.89	3.15
0.25	**3**	4.62	3.53	3.71	1.52	1.84	2.12
	5	5	4.02	4.18	2.17	2.47	2.73
	7	5.26	4.3	4.44	2.54	2.83	3.08
0.50	**3**	4.34	3.34	3.5	1.43	1.73	2
	5	4.65	3.72	3.87	2.01	2.29	2.53
	7	4.81	3.94	4.09	2.33	2.6	2.83
0.75	**3**	3.89	2.98	3.13	1.28	1.55	1.79
	5	3.96	3.18	3.3	1.71	1.95	2.16
	7	4.02	3.29	3.4	1.94	2.16	2.35
1	**3**	3.16	2.4	2.52	1.03	1.25	1.44
	5	2.75	2.2	2.29	1.18	1.35	1.49
	7	2.53	2.06	2.13	1.19	1.33	1.45

Table 2.5. Control Limits Multipliers of *GINI* control chart based on MRSS for different runs rules at $\alpha = 0.0027$

ρ	n	Rules					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0	**3**	4.68	3.59	3.76	1.54	1.87	2.15
	5	2.44	1.97	2.05	1.08	1.23	1.35
	7	2.19	1.81	1.87	1.09	1.21	1.32
0.25	**3**	3.08	2.35	2.47	1.02	1.23	1.41
	5	2.39	1.93	2	1.05	1.2	1.32
	7	2.11	1.76	1.81	1.07	1.18	1.28
0.50	**3**	2.89	2.22	2.33	0.96	1.16	1.33
	5	2.22	1.79	1.86	0.98	1.11	1.23
	7	1.94	1.61	1.66	0.98	1.09	1.18
0.75	**3**	2.59	1.99	2.08	0.86	1.04	1.19
	5	1.88	1.52	1.58	0.83	0.95	1.05
	7	1.62	1.34	1.39	0.82	0.9	0.98
1	**3**	2.11	1.6	1.68	0.69	0.83	0.96
	5	1.31	1.06	1.1	0.58	0.66	0.72
	7	1.02	0.83	0.86	0.5	0.56	0.61

Table 2.6. Control Limits Multipliers of *MAD* control chart based on MRSS for different runs rules at $\alpha = 0.0027$

ρ	*n*	Rules					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0	**3**	4.68	3.59	3.76	1.54	1.87	2.15
	5	1.68	1.28	1.34	0.57	0.68	0.78
	7	1.59	1.26	1.32	0.62	0.72	0.81
0.25	**3**	2.09	1.55	1.63	0.63	0.77	0.89
	5	1.64	1.25	1.31	0.56	0.67	0.76
	7	1.55	1.21	1.26	0.6	0.7	0.78
0.50	**3**	1.95	1.46	1.54	0.6	0.73	0.84
	5	1.53	1.16	1.22	0.52	0.62	0.71
	7	1.42	1.11	1.16	0.55	0.64	0.72
0.75	**3**	1.77	1.31	1.38	0.53	0.65	0.76
	5	1.3	0.99	1.04	0.44	0.53	0.6
	7	1.18	0.93	0.97	0.46	0.53	0.6
1	**3**	1.43	1.06	1.12	0.43	0.52	0.61
	5	0.9	0.69	0.72	0.31	0.36	0.42
	7	0.71	0.57	0.59	0.29	0.33	0.37

Table 2.7. Control Limits Multipliers of $\boldsymbol{Q_n}$ control chart based on MRSS for different runs rules at $\alpha = 0.0027$

ρ	*n*	Rules					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0	**3**	4.68	3.59	3.76	1.54	1.87	2.15
	5	3.47	2.56	2.7	1.07	1.3	1.51
	7	2.83	2.18	2.28	1.1	1.27	1.41
0.25	**3**	4.2	2.91	3.1	0.75	1.05	1.34
	5	3.39	2.5	2.63	1.05	1.27	1.47
	7	2.72	2.11	2.21	1.07	1.24	1.38
0.50	**3**	3.94	2.74	2.92	0.71	0.99	1.26
	5	3.15	2.32	2.45	0.97	1.18	1.36
	7	2.49	1.94	2.02	0.99	1.14	1.27
0.75	**3**	3.56	2.47	2.64	0.64	0.89	1.13
	5	2.68	1.98	2.09	0.83	1.01	1.16
	7	2.06	1.62	1.69	0.82	0.95	1.06
1	**3**	2.89	1.99	2.13	0.51	0.72	0.91
	5	1.85	1.37	1.44	0.57	0.69	0.8
	7	1.33	1	1.05	0.51	0.59	0.65

2.5 Performance evaluation and comparisons

For the proposed runs rules based generalized control charting structure (which merges the variety of sampling strategies and dispersion estimators) we evaluate here the performance of the E_T control charts using power curves as performance criterion. In order to compute the power of the proposed structures we consider shifts in term of $\delta\sigma_X$ which means that the in-control parameter σ_X is shifted by the amount of δ. For power computation we consider the probability limits provided in Tables 2.3-2.7 for specified α, n, m and k using a particular choice of E and T. In our study we have mainly focused on those runs rules schemes and sampling strategies which were recommended by Riaz et al.(2011) and Mehmood et al.(2013b), respectively.

For a fixed false alarm rate α, we choose the control limit coefficient $A_{(T,E_T,n,\rho,m,k,(1-p))}$ from Tables 2.3-2.7, for a given "E", "T", n, ρ, m, k and p. Assuming the standard bivariate normal distribution we calculate the charting statistic E_T with varying δ, n, ρ, m and k. Then we compare this with the control limit coefficient (either it is inside or outside the upper sided limit). The proportion of points falling outside the upper signaling limit $A_{(T,E_T,n,\rho,m,k,(1-p))}$ is finally computed for $\delta = 1$ (i.e. no shift in σ_X) and $\delta \neq 1$ (indicating a shift in process σ_X). This task is accomplished through extensive Monte Carlo simulations (10^5 to 10^6 depending on stability of the output). For some selective values of the design parameters n, ρ, m, k and α at varying values of δ power curves are provided in Figures 2.1-2.5 for different choices of E and T.

Figure 2.1. $MRSS, S, Rule 1/1 and\ n = 5 at\ \alpha = 0.0027$

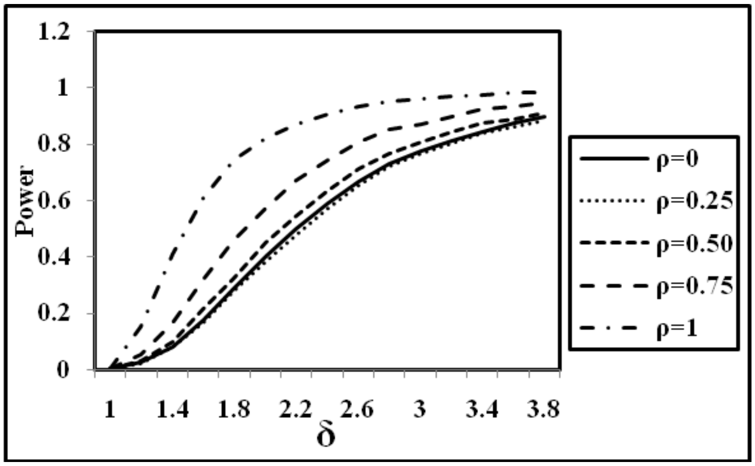


Figure 2.2. $MDRSS, R, Rule\ \ 2/4$ and $\rho = 0.75\ at\ \alpha = 0.0027$.

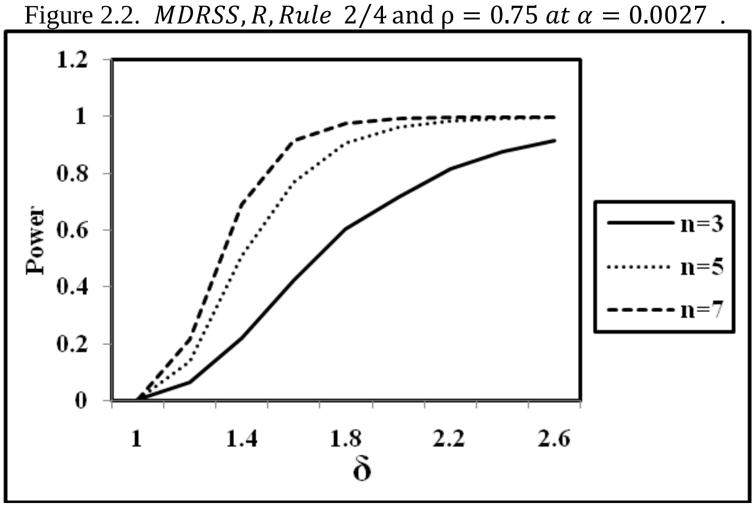


Figure 2.3. $\boldsymbol{DMRSS, GINI}, \mathbf{n = 3, \rho = 0.50}\ \boldsymbol{at\ \alpha = 0.0027}$

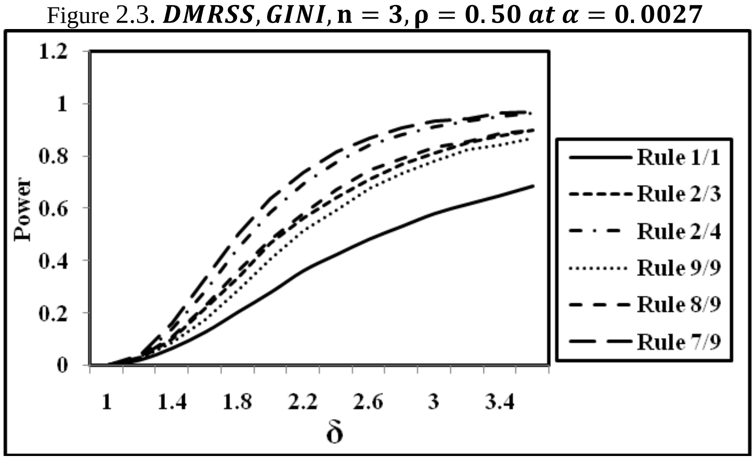


Figure 2. 4. $GINI, Rule\ 1/1$ n $= 5, \rho = 1\ at\ \alpha = 0.0027$

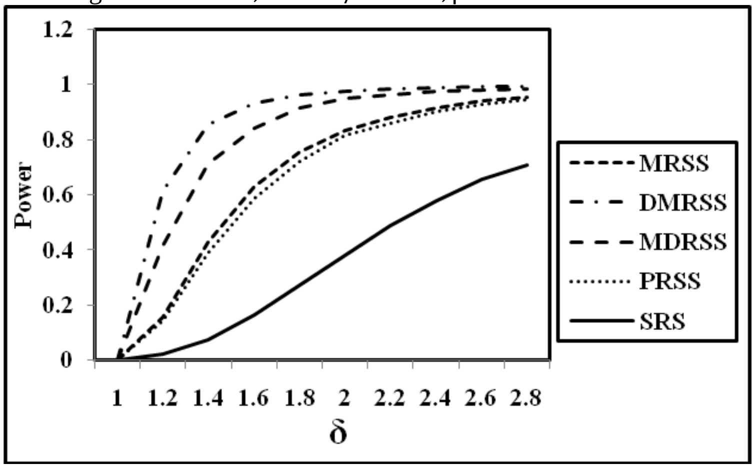
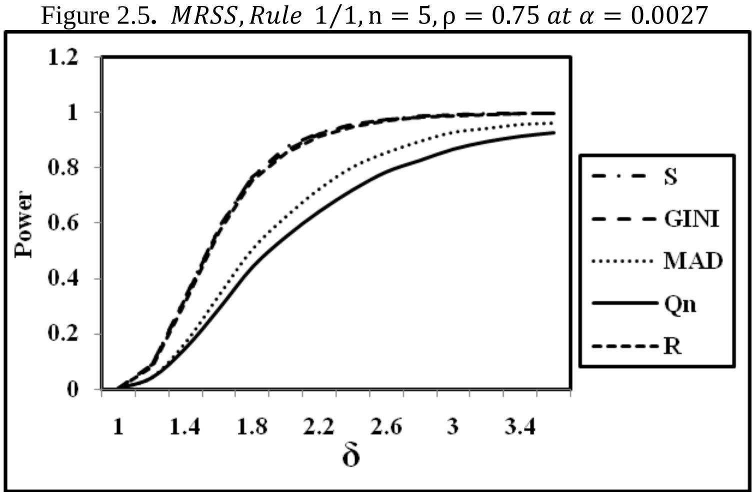


Figure 2.5. $MRSS, Rule\ 1/1,$ n $= 5, \rho = 0.75\ at\ \alpha = 0.0027$



The power analysis reveals that the design structures of the proposed charts are simpler and more accommodative and keep improving their performance with the increase in different quantities including ρ, n, k, $k-m$, δ and α. The proposed schemes outperform the other counterparts including the SRS based design structure, the usual $S, R, GINI, MAD\ and\ Q_n$ charts and the runs rules based design structure of the $S\ and\ R$ charts given by Riaz et al. (2011). With respect to sampling methodologies: double ranked set strategies perform better than the single ranked set strategies in general. In the class of single ranked set strategies the superiority order is MRSS, PRSS, SRS, while in the double ranked set group the dominance order is DMRSS, MDRSS with varying runs rules schemes (cf. Figures 2.1-2.5).

The performance order of different E_T control charts under varying runs rules schemes is as: for small sample sizes S_T, $GINI_T$, and R_T charts are close competitors, while MAD_T and Q_{n_T} charts perform relatively bad; for moderate and larger sample sizes the design structures of S_T and $GINI_T$ charts exhibit the best performance, while R_T, MAD_T and Q_{n_T} present relatively low detection abilities (cf. Figure 2.5) for the process environments under consideration in this study. In general, the implementation of runs rules with different variants of RSS and DRSS strategies enhance the detection ability of these control charting structures (cf. Figure 2.3).

In brief, the proposed E_T control charting structures offer more generalized versions of the usual $S, R, GINI, MAD\, and\, Q_n$ charts (cf. Shewhart, 1931; Schoonhoven et al., 2011; Abbasi & Miller, 2012) , and the runs rules based design structure of the $S\, and\, R$ chart (cf. Riaz et al., 2011). The choices of ρ =0 and/or $m = k$ =1 lead to the said special cases.

2.6 Illustrative example

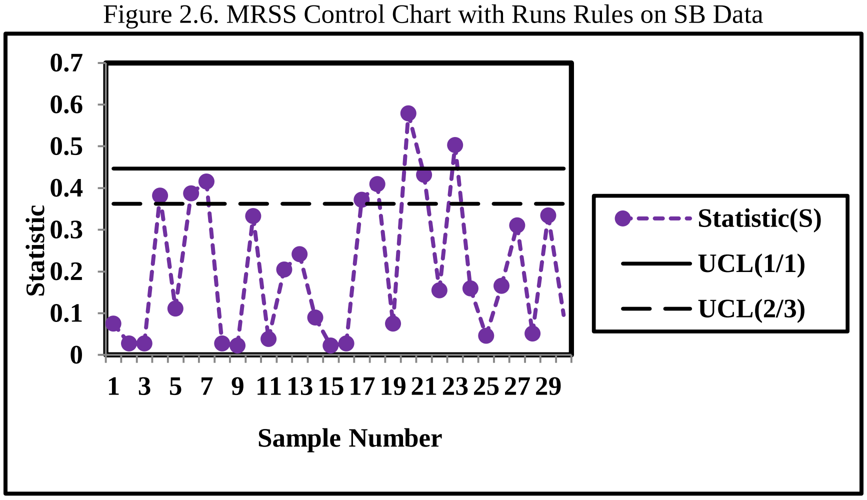
The application of quality control charts can be seen in different disciplines, especially in industry, agriculture and healthcare In this section, we implement our proposed E_T control charts to monitor the health conditions of patients who suffered Hepatitis. The patients are admitted in wards and are treated using different drugs/vaccination. We need to identify whether certain patients may be discharged or still need some treatment. The severity of Hepatitis can be determined through Serum Bilirubin (SB). The optimum level of SB is essential for healthy person. The level of SB can be inspected visually through examination of eyes (sclera). Hence, in our study the variable of interest is SB and the visual inspection of jaundice is our concomitant variable. We obtained a sample from two hospitals (Sheikh Zayed medical college and Bahawalpur Victoria hospital Pakistan) selecting 15 wards from each and then randomly selecting 25 patients (who were under vaccination) from each ward. We divided these patients into 5 groups, each of size 5, and then implemented the MRSS procedure to obtain the median ranked set sample of five patients. After selecting the patients using MRSS we took blood samples of each patient and then tested their SB(mg/dm) in the laboratory. This procedure was done repeatedly and finally we attained 30 MRSS of patients each of size 5.

Using the collected information under MRSS, we constructed the proposed control charts under discussion in order to monitor large as well as small to moderate shifts in SB. Moreover, the summary of the final output is also provided which contains the information of the control chart, the sampling strategy, the sample size, the false alarm rate, the rule type, the control limit (estimated), the sample number where the out-of-control signals detected, and the total out-of-control signals. They are given below. The control charts for different dispersion charts under varying runs rules schemes, and the summary measures are obtained through a code written in R language, following Riaz et al. (2011), Mehmood et al. (2013a), and Mehmood et al. (2013b). We present here two selective runs rules (i.e. Rule1/1 and Rule 2/3) for the *S* control chart based on MRSS in the form of summary measures and a graphical display (cf. Figure 2.6) for examination purposes. The other charts may be constructed in the same way.

SUMMARY OF THE FINAL OUTPUT OF SB DATA
Control Chart : S
Sampling Strategy: MRSS
Sample Size : 5
False Alarm Rate: 0.0027
Rule Type : 1 / 1
Control Limit (U): 0.8519244
Out of Control Signal Received at sample Number: 20 23
Total Out Of Control Signal: 2

SUMMARY OF THE FINAL OUTPUT of SB DATA
Control Chart : S
Sampling Strategy: MRSS
Sample Size : 5
False Alarm Rate: 0.0027
Rule Type : 2 / 3
ontrol Limit (U): 0.6830773
Out of Control Signal Received at sample Number: 6 7 18 19 20 21 22 23
Total Out Of Control Signal: 8

Figure 2.6. MRSS Control Chart with Runs Rules on SB Data



It is apparent that **two** out-of-control signals are received using 1/1 rule and **eight** out-of-control signals are received by 2/3 rule. By implementing efficient runs rules we may be better able to find out the patients with slightly higher SB levels. These results show the importance of the proposed E_T control charting structures in healthcare (one may see similar applications in other fields). These results are in accordance with our findings of Section 2.5.

2.7 Summary and conclusions

This chapter has proposed a generalized version of dispersion control charts based on a variety of sampling strategies, dispersion estimators and runs rules schemes. Among different sampling strategies, DMRSS outperforms the others followed by MDRSS and then comes the single ranked set strategies MRSS and PRSS in the said order. The performance order of different estimators under both single and double ranked set strategies shows that *S* and *GINI* stay on superior end while *R*, *MAD* and Q_n charting structures offer relatively lower performance under different ranked set strategies with varying runs rules schemes. The special cases of the proposed schemes include the usual SRS (e.g. the usual Shewhart *S*, *R*, *GINI*, *MAD*, and Q_n charts (cf. Shewhart, 1931; Schoonhoven et al., 2011; and, Abbasi & Miller, 2012) and runs rules based design structure of *R* and *S* charts (cf. Riaz et al., 2011).

CHAPTER 3

ON EFFICIENT SKEWNESS CORRECTION CHARTS UNDER CONTAMINATION AND NON-NORMALITY

Shewhart control charts are very popular in a variety of disciplines such as industrial process, agriculture and medical science. The design structure of the usual Shewhart charts depends on normality and one point decision rule. This makes the scope of these charts quite limited and not very efficient for small shifts. This chapter comes up with an intermediate solution by implementing runs rules schemes and adjusting the limits' coefficients for non-normality using the idea of skewness correction. We have covered some commonly used location and dispersion charts namely $\bar{X}$, R and S charts. We have investigated the performance of the proposed schemes in terms of false alarm rate and signaling probability. We have observed that the proposed schemes serve the dual purpose in terms of robustness and efficiency. The current chapter also provides an application example using numerical dataset.

3.1 Introduction

Every process consists of two sources of variation termed as natural and un-natural. These variations may have association with either location or dispersion parameter of the process. For the monitoring of these variations, Shewhart control charts are quite popular in literature (cf. Shewhart, 1931). The commonly used Shewhart control charts include R, S and $\bar{X}$ charts that work together to monitor location and dispersions parameters. The usual structures of these charts depend on the assumption of normality and one point decision rule i.e 1/1 (cf. Montgomery, 2009). In case of non-normality or more runs rules schemes, we face the problem of inflated false alarm rates unless handled carefully. Chan & Cui (2003) recommended skewness correction (SC) control charts for location and dispersion by considering the R and $\bar{X}$ charts. They proposed adjustment in the control limits coefficients by considering the level of skewness to relax the strict normality assumption. They used 1/1 rule to take decision for out-of-control signals. The advantage of skewness correction control charts was to develop more general design structures which depend on the skewness of the variable of interests instead of usual assumption of normality that assumes no skewness. Riaz et al. (2011) proposed a variety of runs rules schemes that have independent identities. They covered R, S, and S^2 charts under normality assumption and claimed improved detection ability with these new schemes. This chapter is planned to merge the ideas of Chan & Cui (2003) and Riaz et al. (2011).

In this chapter, we propose three runs rules based skewness correction control charts for the monitoring of location and dispersion parameter using $\bar{X}$, R and S charts. Rest of the chapter is organized as follows: Section 3.2 presents three runs rules based skewness correction control charts for the monitoring of location and dispersion parameters;

Section 3.3 provides performance evaluations and comparisons in terms of false alarms and signaling probabilities; Section 3.4 uses numerical data set to exemplify the practical importance of proposed control charts; Section 3.5 includes summary of the whole study with conclusive remarks.

3.2 Skewness correction control charts with runs rules

In this section, we provide the description of the commonly used $\bar{X}$, R and S charts under two setups namely: the usual Shewhart setup (USH); the skewness correction (SC) of Chan & Cui (2003). Based on these setups we propose runs rules based skewness correction (RRSC) charting structures.

Let $x_1, x_2, ..., x_n$ be a random sample of size *n* (for a quality characteristic of interest say *X*) from a process with mean μ_X and standard deviation σ_X . The control limits for $\bar{X}$, R and S charts for the three setups mentioned above are given as:

USH : Assuming α as the false alarm rate, control limits for the three usual Shewhart control charts under consideration in this study are give below. We have used H_L and H_U to refer to lower and upper limits respectively for a control chart.

$\bar{X}_{USH}$ Chart: $H_{L(\bar{X})} = \mu_{\bar{X}} + (Z_{(\frac{\alpha}{2})})\sigma_{\bar{X}}$ and $H_{U(\bar{X})} = \mu_{\bar{X}} + (Z_{(1-\frac{\alpha}{2})})\sigma_{\bar{X}}$,

R_{USH} Chart: $H_{L(R)} = \mu_R + (Z_{(\frac{\alpha}{2})})\sigma_R$ and $H_{U(R)} = \mu_R + (Z_{(1-\frac{\alpha}{2})})\sigma_R$.

S_{USH} Chart: $H_{L(S)} = \mu_S + (Z_{(\frac{\alpha}{2})})\sigma_S$ and $H_{U(S)} = \mu_S + (Z_{(1-\frac{\alpha}{2})})\sigma_S$, where $Z_{(\frac{\alpha}{2})}$ and $Z_{(1-\frac{\alpha}{2})}$ are $(\frac{\alpha}{2})th$ and $(1-\frac{\alpha}{2})th$ quantiles of standard normal distribution, $\mu_{\bar{X}}$ and $\sigma_{\bar{X}}$ are mean and standard deviation of the sampling distribution of $\bar{X}$, μ_R and σ_R are mean and standard deviation of the sampling distribution of *R* and μ_S and σ_S are mean and standard deviation of the sampling distribution of *S*.

These structures are the generalized versions of 3-sigma limits of $\bar{X}$, *R* and *S* charts. The use of quantiles from standard normal distribution will maintain the false alarm rate of $\bar{X}$ chart at α level, however this will not be the case with the other two charts. These two charts need an adjustment in setting the quantile in order to maintain the desired target of α. An alternative may be to use probability limits for these charts. Riaz et al. (2011) implemented the runs rules based schemes for these charts with probability limits approach under a known parameters case. They defined the runs rules as follows: "any process can be declared as out-of-control if at least *k*-*m* out of *k* consecutive points (where, $0 \le m \le k - 1$) falls outside the limits determined by the sampling distribution of the statistic (like, $\bar{X}$, R, S)". Now the above three structures of $\bar{X}$, R and S control charts under runs rules are as follow:

$\bar{X}_{USH}$ Chart: $H_{L(\bar{X})} = \mu_{\bar{X}} + (Z_{(k,m,\frac{p}{2})})\sigma_{\bar{X}}$ and $H_{U(\bar{X})} = \mu_{\bar{X}} + (Z_{(k,m,1-\frac{p}{2})})\sigma_{\bar{X}}$,

R_{USH} Chart: $H_{L(R)} = \mu_R + (Z_{(k,m,\frac{p}{2})})\sigma_R$ and $H_{U(R)} = \mu_R + (Z_{(k,m,1-\frac{p}{2})})\sigma_R$,

S_{USH} Chart: $H_{L(S)} = \mu_S + (Z_{(k,m,\frac{p}{2})})\sigma_S$ and $H_{U(S)} = \mu_S + (Z_{(1-\frac{p}{2})})\sigma_S$.

where, k refers to the total consecutive points, k-m denotes the decision observations in order to declare out-of-control and p represents the probability of single point falling outside the limits. The value of p depends on the value of k, m and α, and may be obtained by solving the equation $\alpha = \sum_{k-m \le k} \frac{k!}{(k-m)!m!} p^{(k-m)}(1-p)^m$, *where* $0 \le m \le k-1$. The value of p is tabulated in Table 2.1 (see, Chapter 2) for some selective choices of k and m at $\alpha = 0.0027$. The multipliers $Z_{(k,m,\frac{p}{2})}$ and $Z_{(k,m,1-\frac{p}{2})}$ are $(\frac{p}{2})th$ and $(1-\frac{p}{2})th$ quantiles of standard normal distribution for specified values of k, m and α.

In case of unknown parameters ($\mu_{\bar{X}}, \sigma_{\bar{X}}$ and σ_X), the estimated form of USH structures are given as:

$$\bar{X}_{USH} \text{ Chart: } H_{L(\bar{X})} = \bar{\bar{X}} + \left[Z_{(k,m,\frac{p}{2})} \right] \frac{\bar{R}}{d_2\sqrt{n}} = \bar{\bar{X}} + B_L \bar{R} \text{ and } H_{U(\bar{X})} = \bar{\bar{X}} + \left[Z_{(k,m,1-\frac{p}{2})} \right] \frac{\bar{R}}{d_2\sqrt{n}} = \bar{\bar{X}} + B_U \bar{R}, \tag{3.1}$$

$$R_{USH} \text{ Chart: } H_{L(R)} = \left[1 + (Z_{(k,m,\frac{p}{2})}) \frac{d_3}{d_2} \right] \bar{R} = D_L \bar{R} \text{ and } H_{U(R)} = \left[1 + \left(Z_{(k,m,1-\frac{p}{2})} \right) \frac{d_3}{d_2} \right] \bar{R} = D_U \bar{R}, \tag{3.2}$$

$$S_{USH} \text{ Chart: } H_{L(S)} = \left[1 + (Z_{(k,m,\frac{p}{2})}) \frac{e_3}{e_2} \right] \bar{S} = E_L \bar{S} \text{ and } H_{U(S)} = \left[1 + (Z_{(k,m,\frac{p}{2})}) \frac{e_3}{e_2} \right] \bar{S} = E_U \bar{S}, \tag{3.3}$$

where
$$B_L = (Z_{(k,m,\frac{p}{2})}) / d_2\sqrt{n},\ B_U = (Z_{(k,m,1-\frac{p}{2})}) / d_2\sqrt{n},$$
$$D_L = [1 + (Z_{(k,m,\frac{p}{2})}) \frac{d_3}{d_2}],\ D_U = [1 + (Z_{(k,m,1-\frac{p}{2})}) \frac{d_3}{d_2}],$$
$$E_L = [1 + (Z_{(k,m,\frac{p}{2})}) \frac{e_3}{e_2}] \text{ and } E_U = [1 + (Z_{(k,m,1-\frac{p}{2})}) \frac{e_3}{e_2}].$$

Moreover, $\bar{\bar{X}}$, $\bar{R}$ and $\bar{S}$ are means of the respective sample statistics $\bar{X}$, R and S based on an appropriate number of initial in-control samples (say 30-50). The constants d_2 and d_3 (for R chart) are the mean and standard deviation of the relative range. Similarly the constants e_2 and e_3 (for *S* chart) are mean and standard deviation of the relative standard deviation. These constants depend on sample size (*n*) with underlying normal distribution. The detailed description of these constant may be seen in Montgomery (2009). It is to be noted that the control limits coefficients ($Z_{(k,m,\frac{p}{2})}$ and $Z_{(k,m,1-\frac{p}{2})}$) need an adjustment for the estimation effect in order to maintain α at the desired level (cf. Schoonhoven et al., 2011; and Albers & Kallenberg, 2004).

SC : Chan & Cui (2003) proposed *skewness correction* based control limits for *R* and $\bar{X}$ charts. They considered the idea of Cornish-Fisher expansion to set the control limits of a control chart. They picked the quantiles of normal distribution and introduced skewness correction to incorporate the effect of skewness on the limits to maintain the desired false alarm rate. They covered Weibull, Burr and Lognormal distributions for the said purposes. They used a fixed multiplier '3' in the limits of *R* and $\bar{X}$ charts. We provide here a general version by replacing '3' by an α^{th} quantile of standard normal distribution. We have also included the structure for *S* chart following the similar lines.

$\bar{X}_{SC}$ Chart: $H_{L(\bar{X})} = \mu_{\bar{X}} + (Z_{(\frac{\alpha}{2})} + c_4^*)\sigma_{\bar{X}}$ and $H_{U(\bar{X})} = \mu_{\bar{X}} + (Z_{(1-\frac{\alpha}{2})} + c_4^*)\sigma_{\bar{X}}$,

R_{SC} Chart: $H_{L(R)} = \mu_R + (Z_{(\frac{\alpha}{2})} + d_4^*)\sigma_R$ and $H_{U(R)} = \mu_R + (Z_{(1-\frac{\alpha}{2})} + d_4^*)\sigma_R$

S_{SC} Chart: $H_{L(S)} = \mu_S + (Z_{(\frac{\alpha}{2})} + e_4^*)\sigma_S$ and $H_{U(S)} = \mu_S + (Z_{(1-\frac{\alpha}{2})} + e_4^*)\sigma_S$

where, ${c^*}_4$, d_4^* and e_4^* are skewness adjustment factors and defined as:

$$c_4^* = \frac{\frac{1}{6}(Z^2_{(\frac{\alpha}{2})} - 1)k_3(\bar{X})}{1 + 0.2k_3^2(\bar{X})} = \frac{\frac{1}{6}(Z^2_{(1-\frac{\alpha}{2})} - 1)k_3(\bar{X})}{1 + 0.2k_3^2(\bar{X})}, \; d_4^* = \frac{\frac{1}{6}(Z^2_{(\frac{\alpha}{2})} - 1)k_3(R)}{1 + 0.2k_3^2(R)} = \frac{\frac{1}{6}(Z^2_{(1-\frac{\alpha}{2})} - 1)k_3(R)}{1 + 0.2k_3^2(R)}$$

$$, \; e_4^* = \frac{\frac{1}{6}(Z^2_{(\frac{\alpha}{2})} - 1)k_3(S)}{1 + 0.2k_3^2(S)} = \frac{\frac{1}{6}(Z^2_{(1-\frac{\alpha}{2})} - 1)k_3(S)}{1 + 0.2k_3^2(S)},$$

where, $k_3(\bar{X})$, $k_3(R)$ and $k_3(S)$ are the skewness of sample mean $\bar{X}$, relative range $\frac{R}{\sigma}$ and relative standard deviation $\frac{S}{\sigma}$ respectively.

In case of unknown parameters, the estimated form of SC structures are given as:

$\bar{X}_{SC}$ Chart: $H_{L(\bar{X})} = \bar{\bar{X}} + \left[Z_{(\frac{\alpha}{2})} + c_4^* \right] \frac{\bar{R}}{d_2^* \sqrt{n}} = \bar{\bar{X}} + B_L^* \bar{R}$ and $H_{U(\bar{X})} = \bar{\bar{X}} + \left[Z_{(1-\frac{\alpha}{2})} + c_4^* \right] \frac{\bar{R}}{d_2^* \sqrt{n}} = \bar{\bar{X}} + B_U^* \bar{R}$, (3.4)

R_{SC} Chart: $H_{L(R)} = \left[1 + \left(Z_{(\frac{\alpha}{2})} + d_4^* \right) \frac{d_3^*}{d_2^*} \right] \bar{R} = D_L^* \bar{R}$ and $H_{U(R)} = \left[1 + \left(Z_{(1-\frac{\alpha}{2})} + d_4^* \right) \frac{d_3^*}{d_2^*} \right] \bar{R} = D_U^* \bar{R}$, (3.5)

S_{SC} Chart: $H_{L(S)} = \left[1 + \left(Z_{(\frac{\alpha}{2})} + e_4^* \right) \frac{e_3^*}{e_2^*} \right] \bar{S} = E_L^* \bar{S}$ and $H_{U(S)} = \left[1 + \left(Z_{(1-\frac{\alpha}{2})} + e_4^* \right) \frac{e_3^*}{e_2^*} \right] \bar{S} = E_U^* \bar{S}$, (3.6)

where $B_L^* = (Z_{(\frac{\alpha}{2})} + c_4^*) / d_2^* \sqrt{n}$, $B_U^* = (Z_{(1-\frac{\alpha}{2})} + c_4^*) / d_2^* \sqrt{n}$,

$$D_L^* = [1 + (Z_{(\frac{\alpha}{2})} + d_4^*) \frac{d_3^*}{d_2^*}], \; D_U^* = [1 + (Z_{(1-\frac{\alpha}{2})} + d_4^*) \frac{d_3^*}{d_2^*}],$$

$$E_L^* = [1 + (Z_{(\frac{\alpha}{2})} + e_4^*) \frac{e_3^*}{e_2^*}] \text{ and } E_U^* = [1 + (Z_{(1-\frac{\alpha}{2})} + e_4^*) \frac{e_3^*}{e_2^*}].$$

The constants d_2^* and d_3^* (for R chart) are the mean and standard deviation of the relative range. Similarly the constants e_2^* and e_3^* (for S chart) are mean and standard deviation of the relative standard deviation. These constants depend on the skewness level (k_3) instead of strict assumption of normality and sample size (n). The detailed description of these constant may be seen in Chan & Cui (2003). The constants e_2^* and e_3^* for S chart (along with d_2^* and d_3^* for R chart) are provided in Table 3.1 for varying choices of n and k_3.

RRSC : In this section we merge the runs rules idea of Riaz et al. (2011) and skewness correction idea of Chan & Cui (2003), and proposed *runs rules based skewness correction* structures for $\bar{X}$, R and S charts as follows:

$\bar{X}_{RRSC}$ Chart: $H_{L(\bar{X})} = \mu_{\bar{X}} + (Z_{(k,m,\frac{p}{2})} + c_4^{**})\sigma_{\bar{X}}$ and $H_{U(\bar{X})} = \mu_{\bar{X}} + (Z_{(k,m,1-\frac{p}{2})} + c_4^{**})\sigma_{\bar{X}}$,

R_{RRSC} Chart: $H_{L(R)} = \mu_R + (Z_{(k,m,\frac{p}{2})} + d_4^{**})\sigma_R$ and $H_{U(R)} = \mu_R + (Z_{(k,m,1-\frac{p}{2})} + d_4^{**})\sigma_R$,

S_{RRSC} Chart: $H_{L(S)} = \mu_S + (Z_{(k,m,\frac{p}{2})} + e_4^{**})\sigma_S$ and $H_{U(S)} = \mu_S + (Z_{(k,m,1-\frac{p}{2})} + e_4^{**})\sigma_S$,

The other quantities ($c^{**}{}_4$, d_4^{**} and e_4^{**}) are defined as:

$$c_4^{**} = \frac{\frac{1}{6}(Z^2_{(k,m,\frac{p}{2})} - 1)k_3(\bar{X})}{1 - 0.2k_3^2(\bar{X})} = \frac{\frac{1}{6}(Z^2_{(k,m,1-\frac{p}{2})} - 1)k_3(\bar{X})}{1 - 0.2k_3^2(\bar{X})}, \quad d_4^{**} = \frac{\frac{1}{6}(Z^2_{(k,m,\frac{p}{2})} - 1)k_3(R)}{1 - 0.2k_3^2(R)} = \frac{\frac{1}{6}(Z^2_{(k,m,1-\frac{p}{2})} - 1)k_3(R)}{1 - 0.2k_3^2(R)},$$

$$e_4^{**} = \frac{\frac{1}{6}(Z^2_{(k,m,\frac{p}{2})} - 1)k_3(S)}{1 - 0.2k_3^2(S)} = \frac{\frac{1}{6}(Z^2_{(k,m,1-\frac{p}{2})} - 1)k_3(S)}{1 - 0.2k_3^2(S)}.$$

The theoretical justification of the above quantities has been provided in Appendix A.

In case of unknown parameters the estimated *RRSC* may be given as:

$\bar{X}_{RRSC}$ Chart: $H_{L(\bar{X})} = \bar{\bar{X}} + \left[Z_{(k,m,\frac{p}{2})} + c_4^{**}\right]\frac{\bar{R}}{d_2^*\sqrt{n}} = \bar{\bar{X}} + B_L^{**}\bar{R}$ and $H_{U(\bar{X})} = \bar{\bar{X}} + \left[Z_{(k,m,1-\frac{p}{2})} + c_4^{**}\right]\frac{\bar{R}}{d_2^*\sqrt{n}} = \bar{\bar{X}} + B_U^{**}\bar{R}$, (3.7)

R_{RRSC} Chart: $H_{L(R)} = \left[1 + \left(Z_{(k,m,\frac{p}{2})} + d_4^{**}\right)\frac{d_3^*}{d_2^*}\right]\bar{R} = D_L^{**}\bar{R}$ and $H_{U(R)} = \left[1 + \left(Z_{(k,m,1-\frac{p}{2})} + d_4^{**}\right)\frac{d_3^*}{d_2^*}\right]\bar{R} = D_U^{**}\bar{R}$, (3.8)

S_{RRSC} Chart: $H_{L(S)} = \left[1 + \left(Z_{(k,m,\frac{p}{2})} + e_4^{**}\right)\frac{e_3^*}{e_2^*}\right]\bar{S} = E_L^{**}\bar{S}$ and $H_{U(S)} = \left[1 + \left(Z_{(k,m,1-\frac{p}{2})} + e_4^{**}\right)\frac{e_3^*}{e_2^*}\right]\bar{S} = E_U^{**}\bar{S}$,(3.9) where $B_L^{**} = (Z_{(k,m,\frac{p}{2})} + c_4^{**}) / d_2^* \sqrt{n}$, $B_U^{**} = (Z_{(k,m,1-\frac{p}{2})} + c_4^{**}) / d_2^* \sqrt{n}$,

$$D_L^{**} = [1 + (Z_{(k,m,\frac{p}{2})} + d_4^{**}) \frac{d_3^*}{d_2^*}],\ D_U^{**} = [1 + (Z_{(k,m,1-\frac{p}{2})} + d_4^{**}) \frac{d_3^*}{d_2^*}],$$

$$E_L^{**} = [1 + (Z_{(k,m,\frac{p}{2})} + e_4^{**}) \frac{e_3^*}{e_2^*}] \text{ and } E_U^{**} = [1 + (Z_{(k,m,1-\frac{p}{2})} + e_4^{**}) \frac{e_3^*}{e_2^*}].$$

We have also determined the value of c_4^{**}, d_4^{**} and e_4^{**} depending on values of n, k_3 ,a specific choice of runs rules (k *and* m) and α . In order to compute these constant we have covered a variety of skewed distribution including Weibull, Lognormal and four choices of Burr distribution. Under different choices of skewness level (k_3), parameters of the different distributions are given in Table 3.2. For different choices of skewness level (k_3), distributions (as mentioned above), k, m, n, α we have calculated the quantities c_4^{**}, d_4^{**} and e_4^{**} following Chan & Cui (2003). The results for these constants are provided in Tables 3.3-3.5 for some selective choices of k_3 , k, m and n at α =0.0027. We have also provided the control limit coefficients $B_L^{**}, B_U^{**}, D_L^{**}, D_U^{**}, E_L^{**}$ *and* E_U^{**} in Tables 3.6-3.8. These coefficients for other choices of α, k, m and k_3 can be derived on similar lines. It is to be mentioned that 1/1 for *RRSC* is treated same as *SC* of Chan & Cui (2003).

Table 3.1. Unbiasing Constants of different *RRSC* control charts

k_3	***n***	d_2^*	d_3^*	e_2^*	e_3^*
0.8	**2**	1.1	0.88	0.78	0.62
	3	1.65	0.93	0.87	0.49
	4	2.01	0.94	0.9	0.42
	5	2.27	0.93	0.92	0.38
	7	2.64	0.92	0.94	0.32
	10	3.01	0.9	0.96	0.27
1.20	**2**	1.08	0.91	0.76	0.65
	3	1.61	0.98	0.85	0.52
	4	1.97	1	0.89	0.46
	5	2.23	1.02	0.91	0.41
	7	2.6	1.01	0.93	0.36
	10	2.98	1.02	0.95	0.3
2	**2**	1.01	0.97	0.72	0.7
	3	1.52	1.09	0.8	0.59
	4	1.85	1.14	0.85	0.53
	5	2.11	1.18	0.87	0.49
	7	2.49	1.21	0.9	0.43
	10	2.87	1.25	0.92	0.38
2.40	**2**	0.98	1.02	0.69	0.71
	3	1.48	1.14	0.79	0.62
	4	1.81	1.19	0.82	0.56
	5	2.06	1.24	0.85	0.52
	7	2.44	1.32	0.88	0.47
	10	2.83	1.36	0.91	0.41
4	**2**	0.86	1.07	0.61	0.77
	3	1.32	1.29	0.7	0.72
	4	1.61	1.39	0.75	0.65
	5	1.85	1.43	0.77	0.62
	7	2.23	1.6	0.81	0.57
	10	2.63	1.65	0.85	0.51

Table 3.2. Parameters for different levels of Skewness

Skewness	**Weibull**	**Log Normal**	**Burr**			
	β	σ	**b=3**	**b=4**	**b=6**	**b=7**
0.0	3.60	0.001	7.0	5.8	4.9	4.68
0.4	2.41	0.132	4.4	3.76	3.22	3.085
0.8	1.774	0.257	3.23	2.765	2.375	2.275
1.2	1.398	0.369	2.58	2.215	1.894	1.81
1.6	1.160	0.467	2.196	1.877	1.60	1.525
2.0	1.0	0.55	1.95	1.655	1.40	1.33
2.4	0.886	0.624	1.78	1.50	1.26	1.199
2.8	0.802	0.687	1.66	1.392	1.161	1.1
3.2	0.738	0.742	1.563	1.306	1.085	1.0
3.6	0.688	0.79	1.49305	1.245	1.026	0.97
4.0	0.647	0.833	1.434	1.192	0.977	0.92

Table 3.3. $\boldsymbol{c_4^{**}}$ of $\boldsymbol{\overline{X}_{RRSC}}$ for varying k, m, n, and $\boldsymbol{k_3}$ at $\boldsymbol{\alpha = 0.0027}$

k_3	***n***	**Rules**					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0.8	**2**	0.71	0.32	0.37	-0.05	-0.02	0.01
	3	0.59	0.27	0.31	-0.04	-0.02	0.01
	4	0.52	0.24	0.27	-0.04	-0.02	0.01
	5	0.46	0.21	0.24	-0.03	-0.01	0.01
	7	0.39	0.18	0.21	-0.03	-0.01	0.01
	10	0.33	0.15	0.17	-0.02	-0.01	0.01
1.20	**2**	0.99	0.45	0.52	-0.07	-0.03	0.02
	3	0.84	0.39	0.44	-0.06	-0.03	0.02
	4	0.74	0.34	0.39	-0.05	-0.02	0.02
	5	0.67	0.31	0.36	-0.05	-0.02	0.01
	7	0.58	0.27	0.3	-0.04	-0.02	0.01
	10	0.49	0.22	0.26	-0.04	-0.01	0.01
2	**2**	1.34	0.62	0.71	-0.1	-0.04	0.03
	3	1.21	0.56	0.64	-0.09	-0.04	0.03
	4	1.11	0.51	0.58	-0.08	-0.03	0.02
	5	1.03	0.47	0.54	-0.07	-0.03	0.02
	7	0.9	0.41	0.47	-0.07	-0.03	0.02
	10	0.78	0.36	0.41	-0.06	-0.02	0.02
2.40	**2**	1.43	0.66	0.75	-0.1	-0.04	0.03
	3	1.33	0.61	0.7	-0.1	-0.04	0.03
	4	1.24	0.57	0.65	-0.09	-0.04	0.03
	5	1.16	0.53	0.61	-0.08	-0.03	0.02
	7	1.04	0.48	0.55	-0.07	-0.03	0.02
	10	0.91	0.42	0.48	-0.07	-0.03	0.02
4	**2**	1.45	0.66	0.76	-0.1	-0.04	0.03
	3	1.49	0.68	0.78	-0.11	-0.04	0.03
	4	1.48	0.68	0.78	-0.11	-0.04	0.03
	5	1.45	0.67	0.76	-0.1	-0.04	0.03
	7	1.38	0.63	0.73	-0.1	-0.04	0.03
	10	1.27	0.58	0.67	-0.09	-0.04	0.03

Table 3.4. $\boldsymbol{d_4^{**}}$ of $\boldsymbol{R_{RRSC}}$ for varying k, m, n, and $\boldsymbol{k_3}$at $\boldsymbol{\alpha = 0.0027}$

k_3	n	Rules					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0.8	**2**	1.27	0.58	0.67	-0.09	-0.04	0.03
	3	1.09	0.5	0.57	-0.08	-0.03	0.02
	4	1.01	0.46	0.53	-0.07	-0.03	0.02
	5	0.99	0.46	0.52	-0.07	-0.03	0.02
	7	1	0.46	0.52	-0.07	-0.03	0.02
	10	1.06	0.49	0.56	-0.08	-0.03	0.02
1.20		1.39	0.64	0.73	-0.1	-0.04	0.03
	3	1.26	0.58	0.67	-0.09	-0.04	0.03
	4	1.23	0.56	0.65	-0.09	-0.04	0.03
	5	1.22	0.56	0.64	-0.09	-0.04	0.03
	7	1.21	0.56	0.64	-0.09	-0.04	0.03
	10	1.23	0.56	0.65	-0.09	-0.04	0.03
2	**2**	1.48	0.68	0.78	-0.11	-0.04	0.03
	3	1.47	0.67	0.77	-0.11	-0.04	0.03
	4	1.44	0.66	0.76	-0.1	-0.04	0.03
	5	1.44	0.66	0.76	-0.1	-0.04	0.03
	7	1.42	0.65	0.75	-0.1	-0.04	0.03
	10	1.4	0.64	0.74	-0.1	-0.04	0.03
2.40	**2**	1.45	0.67	0.76	-0.1	-0.04	0.03
	3	1.48	0.68	0.78	-0.11	-0.04	0.03
	4	1.47	0.67	0.77	-0.11	-0.04	0.03
	5	1.46	0.67	0.77	-0.11	-0.04	0.03
	7	1.46	0.67	0.77	-0.11	-0.04	0.03
	10	1.44	0.66	0.76	-0.1	-0.04	0.03
4	**2**	1.28	0.59	0.67	-0.09	-0.04	0.03
	3	1.34	0.62	0.71	-0.1	-0.04	0.03
	4	1.35	0.62	0.71	-0.1	-0.04	0.03
	5	1.43	0.65	0.75	-0.1	-0.04	0.03
	7	1.33	0.61	0.7	-0.1	-0.04	0.03
	10	1.45	0.67	0.76	-0.1	-0.04	0.03

Table 3.5. $\boldsymbol{e_4^{**}}$ of $\boldsymbol{S_{RRSC}}$ for varying *k*, m, n, and $\boldsymbol{k_3}$ at $\boldsymbol{\alpha = 0.0027}$

k_3	***n***	**Rules**					
		1/1	**2/3**	**2/4**	**9/9**	**8/9**	**7/9**
0.8	**2**	1.14	0.52	0.6	-0.08	-0.03	0.02
	3	1.02	0.47	0.54	-0.07	-0.03	0.02
	4	0.99	0.45	0.52	-0.07	-0.03	0.02
	5	0.94	0.43	0.49	-0.07	-0.03	0.02
	7	0.9	0.41	0.47	-0.06	-0.03	0.02
	10	0.86	0.39	0.45	-0.06	-0.03	0.02
1.20	**2**	1.2	0.55	0.63	-0.09	-0.04	0.03
	3	1.14	0.52	0.6	-0.08	-0.03	0.02
	4	1.13	0.52	0.59	-0.08	-0.03	0.02
	5	1.1	0.51	0.58	-0.08	-0.03	0.02
	7	1.07	0.49	0.56	-0.08	-0.03	0.02
	10	1.04	0.48	0.55	-0.07	-0.03	0.02
2	**2**	1.17	0.53	0.61	-0.08	-0.04	0.02
	3	1.18	0.54	0.62	-0.09	-0.04	0.02
	4	1.21	0.55	0.63	-0.09	-0.04	0.03
	5	1.2	0.55	0.63	-0.09	-0.04	0.03
	7	1.18	0.54	0.62	-0.09	-0.04	0.02
	10	1.18	0.54	0.62	-0.09	-0.04	0.02
2.40	**2**	1.13	0.52	0.6	-0.08	-0.03	0.02
	3	1.18	0.54	0.62	-0.09	-0.04	0.02
	4	1.17	0.54	0.62	-0.08	-0.04	0.02
	5	1.19	0.55	0.63	-0.09	-0.04	0.03
	7	1.2	0.55	0.63	-0.09	-0.04	0.03
	10	1.15	0.53	0.6	-0.08	-0.03	0.02
4	**2**	0.93	0.43	0.49	-0.07	-0.03	0.02
	3	0.87	0.4	0.46	-0.06	-0.03	0.02
	4	1.02	0.47	0.54	-0.07	-0.03	0.02
	5	0.93	0.42	0.49	-0.07	-0.03	0.02
	7	1.06	0.49	0.56	-0.08	-0.03	0.02
	10	1.1	0.51	0.58	-0.08	-0.03	0.02

Table 3.6. ($\boldsymbol{B_L^{**}}, \boldsymbol{B_U^{**}}$) of $\boldsymbol{\bar{X}_{RRSC}}$ for varying *n*, *k*, *m*, and $\boldsymbol{k_3}$ at $\boldsymbol{\alpha = 0.0027}$

k_3	**n**	**Rules**											
		1/1		**2/3**		**2/4**		**9/9**		**8/9**		**7/9**	
		$\boldsymbol{B_L}$	$\boldsymbol{B_U}$	$\boldsymbol{B_L}$	$\boldsymbol{B_U}$	$\boldsymbol{B_L}$	$\boldsymbol{B_U}$	$\boldsymbol{B_L}$	$\boldsymbol{B_U}$	$\boldsymbol{B_L}$	$\boldsymbol{B_U}$	$\boldsymbol{B_L}$	$\boldsymbol{B_U}$
0.8	**2**	1.5	2.4	1.2	1.6	1.2	1.7	0.4	0.4	0.6	0.5	0.7	0.7
	3	0.8	1.3	0.7	0.9	0.7	0.9	0.2	0.2	0.3	0.3	0.4	0.4
	4	0.6	0.9	0.5	0.6	0.5	0.6	0.2	0.2	0.2	0.2	0.3	0.3
	5	0.5	0.7	0.4	0.5	0.4	0.5	0.1	0.1	0.2	0.2	0.2	0.2
	7	0.4	0.5	0.3	0.3	0.3	0.4	0.1	0.1	0.1	0.1	0.2	0.2
	10	0.3	0.4	0.2	0.2	0.2	0.3	0.1	0.1	0.1	0.1	0.1	0.1
1.20	**2**	1.3	2.6	1.1	1.7	1.2	1.9	0.5	0.4	0.6	0.6	0.7	0.7
	3	0.8	1.4	0.6	0.9	0.7	1	0.3	0.2	0.3	0.3	0.4	0.4
	4	0.6	1	0.5	0.6	0.5	0.7	0.2	0.2	0.2	0.2	0.3	0.3
	5	0.5	0.7	0.4	0.5	0.4	0.5	0.1	0.1	0.2	0.2	0.2	0.2
	7	0.4	0.5	0.3	0.4	0.3	0.4	0.1	0.1	0.1	0.1	0.2	0.2
	10	0.3	0.4	0.2	0.3	0.2	0.3	0.1	0.1	0.1	0.1	0.1	0.1
2	**2**	1.2	3	1.1	1.9	1.1	2.1	0.5	0.4	0.6	0.6	0.7	0.8
	3	0.7	1.6	0.6	1	0.6	1.1	0.3	0.2	0.4	0.3	0.4	0.4
	4	0.5	1.1	0.4	0.7	0.5	0.8	0.2	0.2	0.2	0.2	0.3	0.3
	5	0.4	0.9	0.4	0.6	0.4	0.6	0.2	0.1	0.2	0.2	0.2	0.2
	7	0.3	0.6	0.3	0.4	0.3	0.4	0.1	0.1	0.1	0.1	0.2	0.2
	10	0.2	0.4	0.2	0.3	0.2	0.3	0.1	0.1	0.1	0.1	0.1	0.1
2.40	**2**	1.1	3.2	1.1	2	1.1	2.2	0.5	0.4	0.7	0.6	0.8	0.8
	3	0.7	1.7	0.6	1.1	0.6	1.2	0.3	0.2	0.4	0.3	0.4	0.4
	4	0.5	1.2	0.4	0.8	0.5	0.8	0.2	0.2	0.3	0.2	0.3	0.3
	5	0.4	0.9	0.4	0.6	0.4	0.6	0.2	0.1	0.2	0.2	0.2	0.2
	7	0.3	0.6	0.3	0.4	0.3	0.4	0.1	0.1	0.1	0.1	0.2	0.2
	10	0.2	0.4	0.2	0.3	0.2	0.3	0.1	0.1	0.1	0.1	0.1	0.1
4	**2**	1.3	3.6	1.2	2.3	1.2	2.5	0.6	0.4	0.7	0.7	0.9	0.9
	3	0.7	2	0.7	1.3	0.7	1.4	0.3	0.2	0.4	0.4	0.5	0.5
	4	0.5	1.4	0.5	0.9	0.5	1	0.2	0.2	0.3	0.3	0.3	0.3
	5	0.4	1.1	0.4	0.7	0.4	0.7	0.2	0.1	0.2	0.2	0.3	0.3
	7	0.3	0.7	0.3	0.5	0.3	0.5	0.1	0.1	0.2	0.1	0.2	0.2
	10	0.2	0.5	0.2	0.3	0.2	0.4	0.1	0.1	0.1	0.1	0.1	0.1

Table 3.7. $(\boldsymbol{D_U^{**}}, \boldsymbol{D_L^{**}})$ $\boldsymbol{o}$f $\boldsymbol{R_{RRSC}}$or varying *n*, *k*, *m*, and $\boldsymbol{k_3}$at $\boldsymbol{\alpha = 0.0027}$

k_3	**n**	**Rules**											
		1/1		**2/3**		**2/4**		**9/9**		**8/9**		**7/9**	
		$\boldsymbol{D_U}$	$\boldsymbol{D_L}$	$\boldsymbol{D_U}$	$\boldsymbol{D_L}$	$\boldsymbol{D_U}$	$\boldsymbol{D_L}$	$\boldsymbol{D_U}$	$\boldsymbol{D_L}$	$\boldsymbol{D_U}$	$\boldsymbol{D_L}$	$\boldsymbol{D_U}$	$\boldsymbol{D_L}$
0.8	**2**	4.4	0	3.2	0	3.4	0	1.5	0.4	1.7	0.3	1.9	0.2
	3	3.3	0	2.5	0.1	2.6	0	1.3	0.6	1.5	0.5	1.6	0.4
	4	2.9	0.1	2.2	0.2	2.3	0.2	1.3	0.7	1.4	0.6	1.5	0.5
	5	2.6	0.2	2.1	0.3	2.2	0.3	1.2	0.7	1.4	0.6	1.5	0.6
	7	2.4	0.3	1.9	0.4	2	0.4	1.2	0.8	1.3	0.7	1.4	0.6
	10	2.2	0.4	1.8	0.5	1.9	0.5	1.2	0.8	1.3	0.7	1.3	0.7
1.20	**2**	4.7	0	3.4	0	3.6	0	1.5	0.4	1.7	0.2	1.9	0.1
	3	3.6	0	2.7	0	2.8	0	1.3	0.6	1.5	0.5	1.7	0.4
	4	3.2	0.1	2.4	0.2	2.5	0.2	1.3	0.6	1.4	0.5	1.6	0.5
	5	2.9	0.2	2.2	0.3	2.3	0.3	1.3	0.7	1.4	0.6	1.5	0.5
	7	2.6	0.3	2.1	0.4	2.1	0.4	1.2	0.7	1.3	0.7	1.4	0.6
	10	2.4	0.4	1.9	0.5	2	0.4	1.2	0.8	1.3	0.7	1.4	0.6
2	**2**	5.3	0	3.7	0	4	0	1.5	0.3	1.8	0.1	2.1	0
	3	4.2	0	3	0	3.2	0	1.4	0.5	1.6	0.4	1.8	0.3
	4	3.7	0.1	2.7	0.1	2.9	0.1	1.3	0.5	1.5	0.4	1.7	0.4
	5	3.5	0.1	2.6	0.2	2.7	0.2	1.3	0.6	1.5	0.5	1.6	0.4
	7	3.2	0.2	2.4	0.3	2.5	0.3	1.3	0.6	1.4	0.6	1.5	0.5
	10	2.9	0.3	2.2	0.3	2.3	0.3	1.2	0.7	1.4	0.6	1.5	0.5
2.40	**2**	5.6	0	3.9	0	4.2	0	1.6	0.2	1.9	0.1	2.2	0
	3	4.4	0	3.2	0	3.4	0	1.4	0.4	1.6	0.3	1.9	0.2
	4	3.9	0	2.9	0	3	0	1.4	0.5	1.5	0.4	1.7	0.3
	5	3.7	0.1	2.7	0.1	2.9	0.1	1.3	0.5	1.5	0.5	1.7	0.4
	7	3.4	0.2	2.5	0.2	2.7	0.2	1.3	0.6	1.5	0.5	1.6	0.4
	10	3.1	0.3	2.4	0.3	2.5	0.3	1.3	0.6	1.4	0.6	1.5	0.5
4	**2**	6.3	0	4.4	0	4.7	0	1.7	0.1	2	0	2.4	0
	3	5.2	0	3.7	0	3.9	0	1.5	0.3	1.8	0.1	2.1	0
	4	4.7	0	3.4	0	3.6	0	1.5	0.4	1.7	0.2	2	0.1
	5	4.4	0	3.2	0	3.4	0	1.4	0.4	1.6	0.3	1.9	0.2
	7	4.1	0	3	0	3.2	0	1.4	0.5	1.6	0.4	1.8	0.2
	10	3.8	0	2.8	0.1	2.9	0	1.3	0.5	1.5	0.4	1.7	0.3

Table 3.8. $(\boldsymbol{E_U^{**}}, \boldsymbol{E_L^{**}})$ of $\boldsymbol{S_{RRSC}}$ for varying *n*, *k*, *m*, and $\boldsymbol{k_3}$ at $\boldsymbol{\alpha = 0.0027}$

k_3	***n***	**Rules**											
		1/1		**2/3**		**2/4**		**9/9**		**8/9**		**7/9**	
		$\boldsymbol{E_U}$	$\boldsymbol{E_L}$	$\boldsymbol{E_U}$	$\boldsymbol{E_L}$	$\boldsymbol{E_U}$	$\boldsymbol{E_L}$	$\boldsymbol{E_U}$	$\boldsymbol{E_L}$	$\boldsymbol{E_U}$	$\boldsymbol{E_L}$	$\boldsymbol{E_U}$	$\boldsymbol{E_L}$
0.8	**2**	4.3	0	3.1	0	3.3	0	1.5	0.4	1.7	0.3	1.9	4.3
	3	3.3	0	2.5	0	2.6	0	1.3	0.6	1.5	0.5	1.6	3.3
	4	2.9	0.1	2.2	0.2	2.3	0.2	1.3	0.7	1.4	0.6	1.5	2.9
	5	2.6	0.2	2.1	0.3	2.1	0.3	1.2	0.7	1.3	0.6	1.5	2.6
	7	2.3	0.3	1.9	0.4	1.9	0.4	1.2	0.8	1.3	0.7	1.4	2.3
	10	2.1	0.4	1.7	0.5	1.8	0.5	1.2	0.8	1.2	0.8	1.3	2.1
1.20	**2**	4.6	0	3.3	0	3.5	0	1.5	0.4	1.7	0.2	1.9	4.6
	3	3.5	0	2.6	0	2.8	0	1.4	0.6	1.5	0.5	1.7	3.5
	4	3.1	0	2.4	0.2	2.5	0.1	1.3	0.6	1.4	0.5	1.6	3.1
	5	2.9	0.2	2.2	0.3	2.3	0.2	1.3	0.7	1.4	0.6	1.5	2.9
	7	2.6	0.3	2	0.4	2.1	0.3	1.2	0.7	1.3	0.7	1.4	2.6
	10	2.3	0.4	1.8	0.5	1.9	0.4	1.2	0.8	1.3	0.7	1.4	2.3
2	**2**	5	0	3.6	0	3.8	0	1.6	0.3	1.8	0.1	2.1	5
	3	4.1	0	3	0	3.1	0	1.4	0.5	1.6	0.3	1.8	4.1
	4	3.6	0	2.7	0	2.8	0	1.4	0.5	1.5	0.4	1.7	3.6
	5	3.3	0	2.5	0.1	2.6	0.1	1.3	0.6	1.5	0.5	1.6	3.3
	7	3	0.1	2.3	0.2	2.4	0.2	1.3	0.7	1.4	0.6	1.5	3
	10	2.7	0.3	2.1	0.3	2.2	0.3	1.2	0.7	1.3	0.6	1.5	2.7
2.40	**2**	5.2	0	3.7	0	4	0	1.6	0.3	1.9	0.1	2.1	5.2
	3	4.3	0	3.1	0	3.3	0	1.4	0.4	1.7	0.3	1.9	4.3
	4	3.8	0	2.8	0	3	0	1.4	0.5	1.6	0.4	1.8	3.8
	5	3.6	0	2.7	0	2.8	0	1.4	0.6	1.5	0.5	1.7	3.6
	7	3.2	0.1	2.4	0.2	2.6	0.1	1.3	0.6	1.4	0.5	1.6	3.2
	10	2.9	0.2	2.2	0.3	2.3	0.2	1.3	0.7	1.4	0.6	1.5	2.9
4	**2**	5.9	0	4.3	0	4.5	0	1.7	0.1	2.1	0	2.4	5.9
	3	5	0	3.6	0	3.8	0	1.6	0.3	1.9	0.1	2.1	5
	4	4.5	0	3.3	0	3.5	0	1.5	0.4	1.7	0.2	2	4.5
	5	4.2	0	3.1	0	3.2	0	1.5	0.4	1.7	0.3	1.9	4.2
	7	3.8	0	2.9	0	3	0	1.4	0.5	1.6	0.4	1.8	3.8
	10	3.5	0	2.6	0	2.8	0	1.4	0.6	1.5	0.5	1.7	3.5

3.3 Performance evaluations and comparisons

In this section, we evaluate and compare the performance of different types of charts under discussion using false alarm rate and signaling probabilities. The false alarm rate is the probability of process breaching outside the signaling limits (i.e. H_L, H_U) when it is in fact in-control. The probability to signal is defined as the detection probability of a given control chart when the processes is actually out-of-control. We have used these signaling probabilities in term of individual measure (named as *Probability To Signals* and abbreviated as *PTS*) as well as overall measure (named as *Average Absolute Difference Of Probability* to signal and denoted as *AADP*)). The *AADP* is defined as: $AADP = (\sum \left| M_{(\delta)} - E_{(\delta)} \right| / t)\, x\, 100$, where δ is the amount of shift in the process parameter (location or dispersion defined in terms of $\delta \sigma_X$), $M_{(\delta)}$ is the probability to signal of the charts under consideration ($\bar{X}_{RRSC}, \bar{X}_{USH}, \bar{X}_{SC}$ $R_{RRSC}, R_{USH}, R_{SC}, S_{RRSC}, S_{USH}$, S_{SC}), $E_{(\delta)}$ represents the probability to signal based on true limits for a known probability distribution (also known as exact distribution based control limits) for a given value δ and *t* is the length of δ. A control chart with smaller *AADP*, keeps the false alarm rate as well as *PTS* close to desired level is considered more desirable.

For our study purposes, we have considered different distributions including skewed (Weibull, Gamma and Chi-square), heavy tailed (Lognormal and different choices of Burr) and contaminated normal. For the case of contaminated normal distribution, we have considered two cases: diffuse mean disturbances (DMD) and diffuse asymmetric variance disturbances (DAVD). Furthermore, in case of DMD, 95% probability of each measured observations being drawn from $N(0,1)$ and 5% probability from $N(a, 1)$, where

'a' is the amount of disturbance. Similarly, in case of DAVD, 95% probability of each measured observation being drawn from $N(0,1)$ and 5% probability of being drawn from $b\chi^2_{(1)}$, which will be further added to $N(0,1)$; where, 'b' is the multiple of chi-square distribution (one degree of freedom). Further details about the above mentioned procedures can be seen in Schoonhoven et al. (2011).

Moreover, we have used different choices of runs rules schemes, sample sizes, skewness level and estimated the control limits (see, Equations 3.1-3.9) for the three types of charts under three setups namely *USH*, *SC* and *RRSC*. For each set of limits we have computed the performance of all the charts under discussion in terms of false alarm rate, *PTS* and *AADP*. The procedure for computing the false alarm rate and *PTS* is given in the following steps.

Step 1: For a given value of k_3, k, m, p, n, a and b, 30 samples of size *n* are generated from a distribution and estimated the control limits given in Equations 3.1-3.9;

Step 2: again 100 samples of size *n* are generated and checked whether the value of test statistic is either inside or outside the estimated control limits.

Step 3: steps 1-2 are repeated forty thousand times and proportion of test statistic is considered as the required false alarm rate of three kinds of charts for a given value of $k_3, k, m, p, n, a, and\, b$.

Step 4: In order to compute probability to signals, the random samples at step 2 are generated by considering the amount of shift in term of $\delta\sigma_X$.

3.3.1 Results and Discussion

We have obtained the results of false alarm rate, *PTS* and *AADP* for varying combinations of k_3, k, m, p, n, a and b . We have followed the above mentioned steps of section 3 to obtain these results and some selective outcomes are provided in Figures 3.1-3.3 and Tables 3.9-3.11. We have also examined the behaviour of the signalling limits (H_L, H_U) for different charts $(\bar{X}_{RRSC}, \bar{X}_{USH}, R_{RRSC}, R_{USH}, S_{RRSC}, S_{USH})$ with varying *n*, *k*, and m at $\alpha = 0.0027$ (see, Figure 3.4). It is to be mentioned that the computational results of this study are obtained using 4×10^4 Monte Carlo simulation.

The results obtained advocate the following for the three types of charting structures under consideration in this study (cf. Figures 3.1-3.4 and Tables 3.9-3.11).

i. The control limits (H_L, H_U) of runs rules based skewness correction control charts (see, Section 3.2) are relatively closer to the exact limits (known distribution case, e.g. here we considered particularly exponential distribution) as compared to the usual Shewhart charts. So, the behavior of runs rules based skewness adjustment control limits (H_L, H_U) offer reasonable robustness (due to the closeness with exact limits), particularly at large *n*. Chan & Cui (2003) gave similar findings for the case when m=0 and k=1(see, Figure 3.4).

ii. The false alarm rate for *USH* charts is relatively more disturbed as compared to *RRSC* at varying values of design parameters including k_3, n, k, m and consequently *PTS* and *AADP* are affected (cf. Figure 3.2-3.3 and Table 3.9-3.11). Note: In Figure 3.3, label ‘Exact’ implies that known probability distribution based (exponential distribution) *PTS.*

iii. *SC (k-m=1,k*=1) charts help in maintaining the false alarm rate for varying $k_3$and *n* at the cost of some loss in signaling probabilities for out-of-control situations (cf. Figure 3.1).

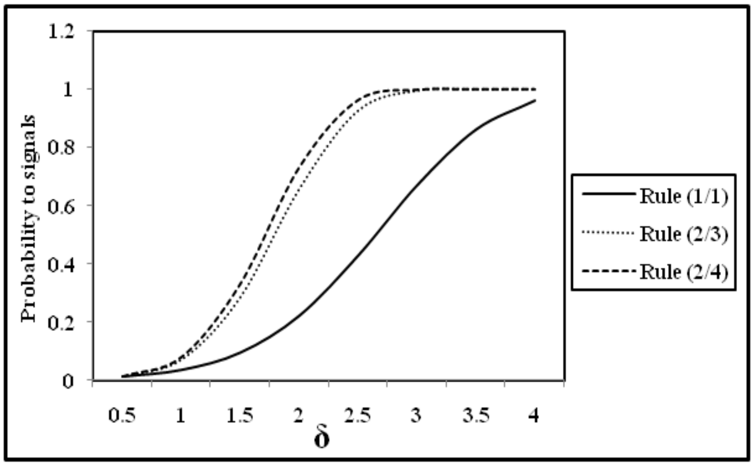
iv. *RRSC* charts help in reasonably maintaining the false alarm rate at varying k_3, *n*, *k and m* as well as increasing the ability towards signaling probabilities (cf. Figure 3.1).

v. For large k_3, it is appropriate to have large sample size (*n*) for heavy tailed distributions in order to achieve better probabilities in term of α and *AADP*. Moreover, implementation of the runs rules schemes (for different choices of *k* and *m*) with varying choices of k_3 and *n* improve the performance of different types of charts in general, For example see the performance of $\bar{X}_{RRSC}, R_{RRSC}$, and S_{RRSC}) and charts. (See Figure 3.1 and Table 3.9-3.11).

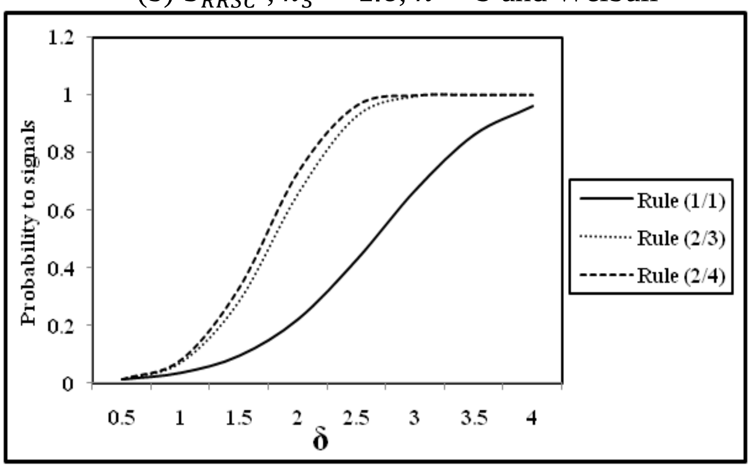
vi. It is interesting to note that $\bar{X}_{RRSC}$ control chart is comprehensive form of the usual runs rules based Shewhart control chart when $k_3 = 0$; the 3-sigma limits based $\bar{X}$control chart when *m*=0, *k*=1 and $k_3 = 0$; skewness correction control ($\bar{X}_{SC}$) when *m*=0 and *k*=1. In addition, *RRSC* control charts for the monitoring of dispersion parameter can accommodate the skewness correction based charts (R_{SC}, and S_{SC}) when *m*=0 and *k*=1.

Figure 3.1. Performance comparison of different runs rules $at\,\alpha = 0.0027$

(a) $\bar{X}_{RRSC}$, $k_3 = 0.8$, n=2 and Burr (2)



(b) S_{RRSC} , $k_3 = 1.6$, $n = 5$ and Weibull



(c) R_{RRSC}, $k_3 = 1.6$, $n = 5$ and Chisquare

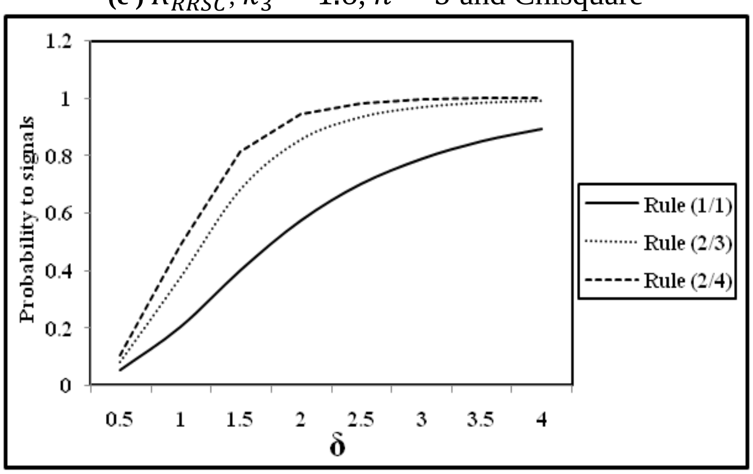
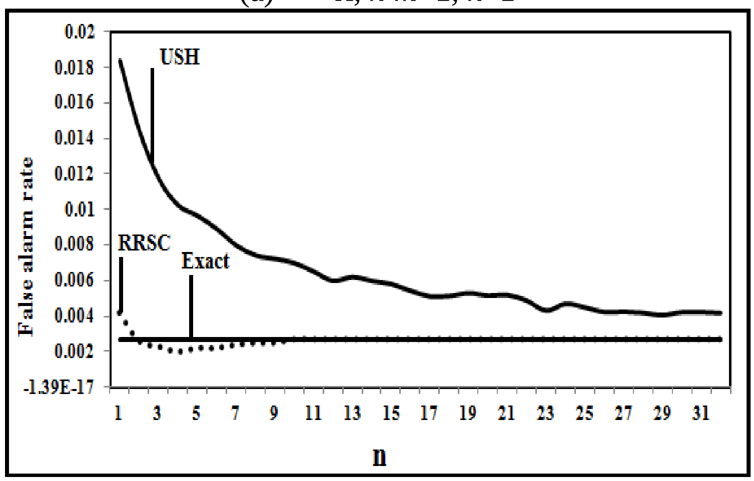
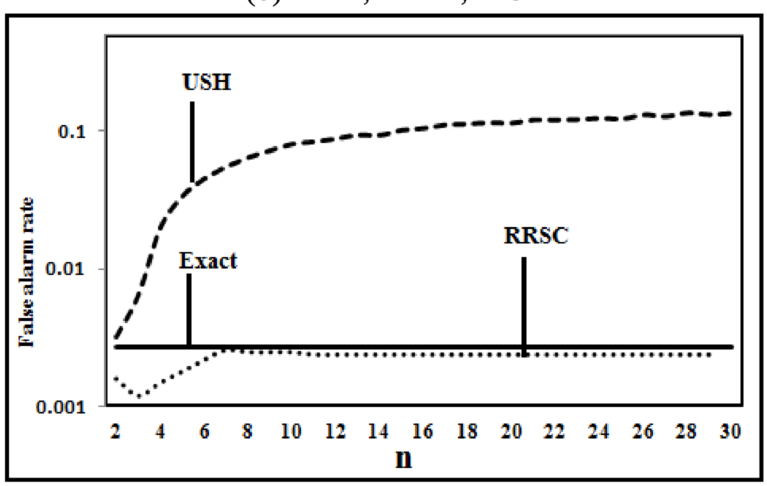


Figure 3.2.False alarm rate comparison of different control charts with varying n, k and m at $\boldsymbol{\alpha = 0.0027}$

(a) $\overline{X}$, k-m=1, k=1



(b) R, k-m=2, k=3



(c) S, k-m=2, k=3

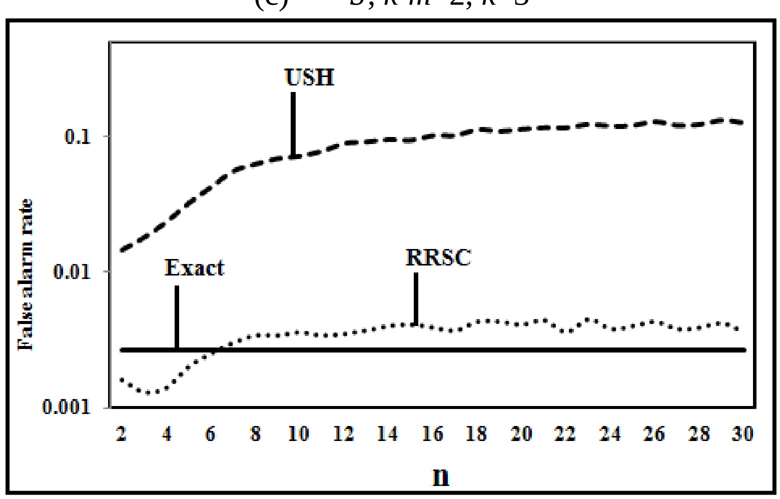
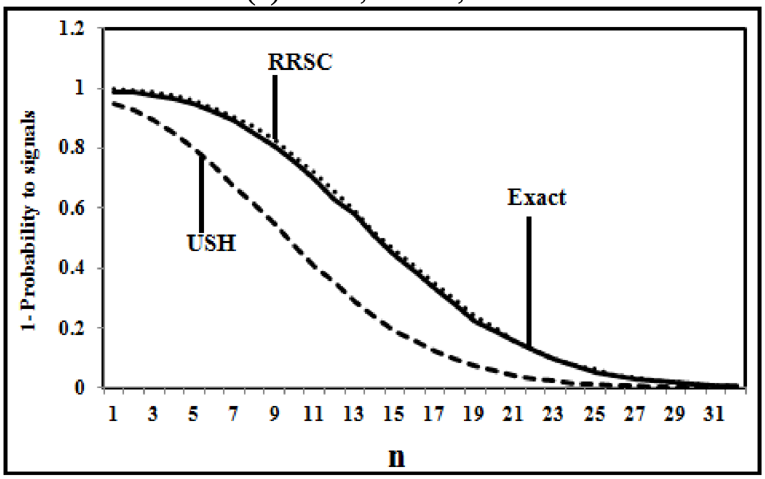
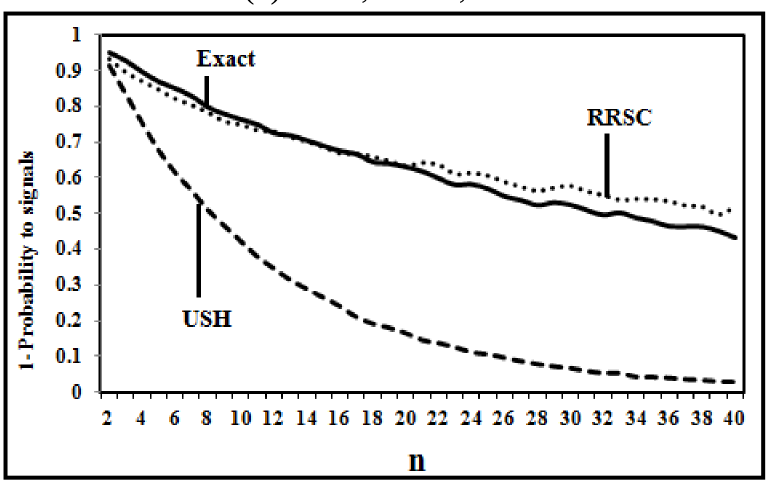


Figure 3.3. *PTS* Comparison of different control charts with varying *n*, *k*, *m*, and $\boldsymbol{\delta = 2}$ at $\boldsymbol{\alpha = 0.0027}$

(a) $\overline{\mathrm{X}}$, *k*-*m*=1, *k*=1



(b) *R*, *k*-*m*=2, *k*=3



(c) *S*, *k*-*m*=2, *k*=3

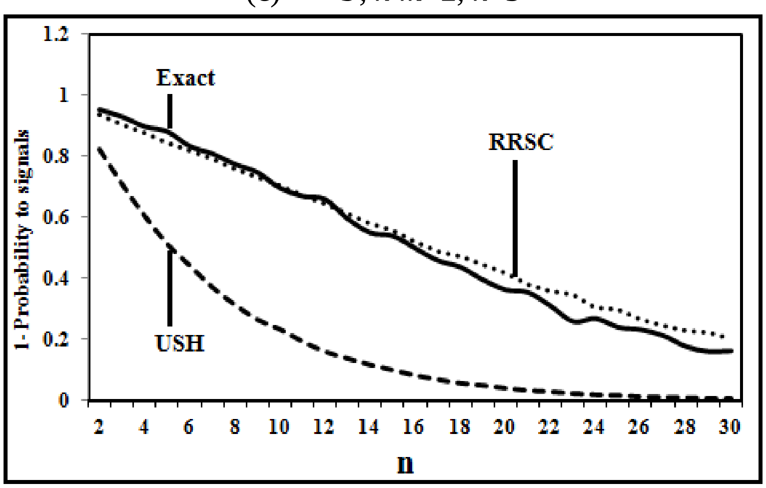
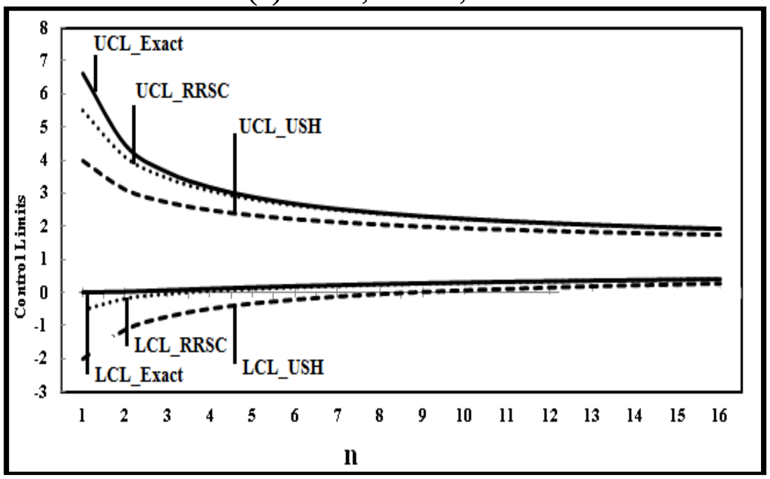
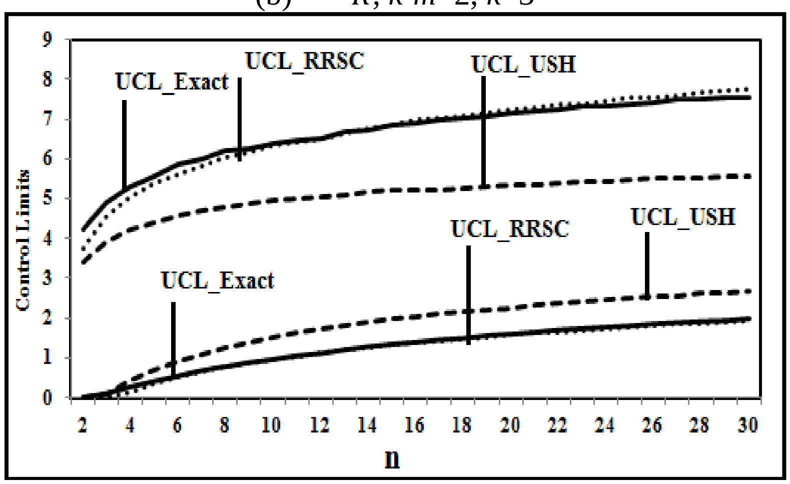


Figure 3.4. Upper and Lower control limits comparison of different control charts with varying n, k and m at $\boldsymbol{\alpha = 0.0027}$

(a) $\overline{\mathrm{X}}$, k-m=1, k=1



(b) R, k-m=2, k=3



(c) S k-m=2, k=3

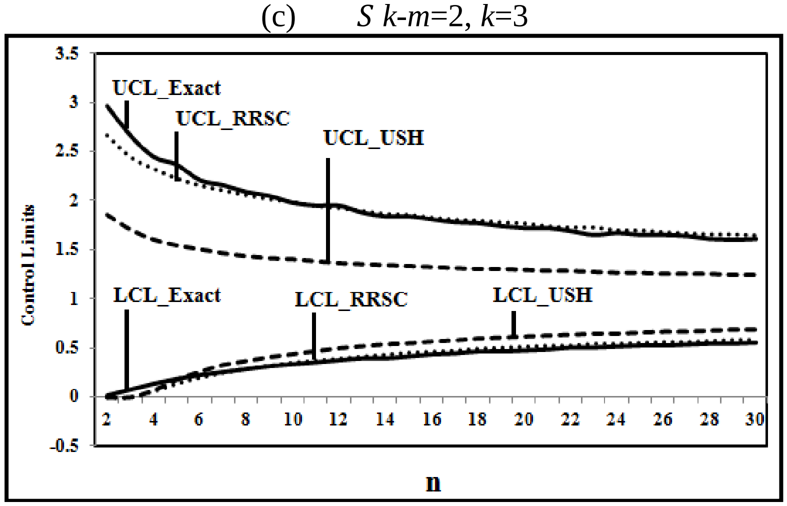


Table 3. 9. False alarm rate of *RRSC* and *USH* control charts with varying *k*, *m and* $\boldsymbol{k_3}$ at $\alpha = 0.0027$

$\bar{X}_{RRSC}$, $\bar{X}_{USH}$, $k-m=1$ and $k=1$								
n	$\boldsymbol{k_3}$	**Chart Type**	**Burr(1)**	**Burr(2)**	**Gamma**	**Chi-square**	**Weibull**	**Lognormal**
5	**0.8**	***RRSC***	0.0047	0.0043	0.0037	0.0037	0.0032	0.004
		USH	0.0066	0.0064	0.0061	0.0061	0.0058	0.0064
	1.6	***RRSC***	0.0064	0.0056	0.0043	0.0043	0.0038	0.0055
		USH	0.012	0.0056	0.0132	0.0043	0.0136	0.013
	2.8	***RRSC***	0.009	0.0078	0.0047	0.0049	0.0052	0.008
		USH	0.0197	0.0209	0.0295	0.0298	0.0279	0.0228
10	**0.8**	***RRSC***	0.0037	0.0036	0.0032	0.0032	0.0035	0.0033
		USH	0.0051	0.0051	0.0047	0.0047	0.0052	0.0049
	1.6	***RRSC***	0.0053	0.005	0.0045	0.0047	0.0045	0.0047
		USH	0.0088	0.005	0.0091	0.0094	0.0095	0.0086
	2.8	***RRSC***	0.0064	0.0059	0.0044	0.0049	0.0048	0.0057
		USH	0.014	0.0153	0.0204	0.0203	0.0196	0.016
R_{RRSC} , R_{USH} , $k-m=2$ and $k=3$								
n	$\boldsymbol{k_3}$	**Chart Type**	**Burr(1)**	**Burr(2)**	**Gamma**	**Chi-square**	**Weibull**	**Lognormal**
5	**0.8**	***RRSC***	0.0036	0.0033	0.0029	0.003	0.0021	0.0033
		USH	0.0085	0.0078	0.007	0.0074	0.0058	0.0085
	1.6	***RRSC***	0.0021	0.0022	0.0028	0.0027	0.003	0.0026
		USH	0.0191	0.0204	0.0272	0.0268	0.0293	0.0225
	2.8	***RRSC***	0.001	0.0011	0.0028	0.0026	0.0034	0.0026
		USH	0.04	0.0495	0.0190	0.0187	0.0185	0.0478
10	**0.8**	***RRSC***	0.0055	0.0045	0.0037	0.0036	0.0022	0.0046
		USH	0.0121	0.0114	0.01	0.0093	0.0062	0.0113
	1.6	***RRSC***	0.0024	0.0024	0.003	0.0028	0.0031	0.0029
		USH	0.0337	0.035	0.045	0.0443	0.0466	0.0421
	2.8	***RRSC***	0.0011	0.0012	0.0032	0.0032	0.0031	0.0029
		USH	0.0846	0.1002	0.0490	0.0496	0.0490	0.1201
S_{RRSC} , S_{USH} , $k-m=2$ and $k=4$								
n	$\boldsymbol{k_3}$	**Chart Type**	**Burr(1)**	**Burr(2)**	**Gamma**	**Chi-square**	**Weibull**	**Lognormal**
5	**0.8**	***RRSC***	0.0039	0.0035	0.0031	0.0031	0.0022	0.0036
		USH	0.0113	0.0105	0.0101	0.0098	0.0075	0.0105
	1.6	***RRSC***	0.0019	0.0019	0.0022	0.0022	0.0021	0.0023
		USH	0.028	0.0297	0.0442	0.0442	0.0439	0.0332
	2.8	***RRSC***	0.0016	0.0015	0.0042	0.0042	0.0037	0.0026
		USH	0.1288	0.0543	0.212	0.212	0.1685	0.0865
10	**0.8**	***RRSC***	0.0052	0.0044	0.0035	0.0036	0.0023	0.0046
		USH	0.0169	0.0152	0.0124	0.0129	0.009	0.0161
	1.6	***RRSC***	0.0025	0.0025	0.003	0.003	0.0029	0.003
		USH	0.0519	0.0553	0.067	0.067	0.0756	0.065
	2.8	***RRSC***	0.0017	0.0014	0.0034	0.0034	0.0029	0.0023
		USH	0.0601	0.1124	0.3054	0.3054	0.2777	0.1709

Table 3.10. False alarm rate of *RRSC* and *USH* control charts under two contaminated case with varying *k, m* and $\boldsymbol{k_3}$ at $\alpha = 0.0027$

Contamination under DMD								
			$\bar{X}_{RRSC}$		R_{RRSC}		S_{RRSC}	
a	k_3		*k-m=1,k=1*		*k-m=2,k=3*		*k-m=2,k=4*	
		n	*RRSC*	*USH*	*RRSC*	*USH*	*RRSC*	*USH*
3.28	0.80	2	0.0113	0.0121	0.0025	0.0004	0.0037	0.0007
		3	0.0068	0.0074	0.0028	0.0004	0.0029	0.0007
		4	0.0062	0.0075	0.0056	0.0006	0.0059	0.0017
		5	0.0049	0.0057	0.0063	0.0004	0.0071	0.0017
		7	0.0040	0.0049	0.0079	0.0005	0.0112	0.0023
		10	0.0028	0.0040	0.0096	0.0006	0.0174	0.0046
4.07	1.20	2	0.0152	0.0167	0.0018	0.0009	0.0049	0.0015
		3	0.0094	0.0104	0.0029	0.001	0.0032	0.0018
		4	0.0077	0.010	0.0055	0.0016	0.0046	0.0044
		5	0.0064	0.0076	0.0048	0.0012	0.0046	0.0045
		7	0.0039	0.0062	0.0058	0.0013	0.0055	0.0057
		10	0.0026	0.0048	0.0052	0.0014	0.0096	0.0101
5.9	2	2	0.0214	0.0346	0.0064	0.0045	0.0109	0.0079
		3	0.0133	0.0174	0.0038	0.0066	0.0045	0.012
		4	0.0105	0.0169	0.0034	0.0104	0.0033	0.0265
		5	0.0071	0.0142	0.0013	0.0083	0.0010	0.0258
		7	0.0045	0.0107	0.0007	0.0077	0.0004	0.0295
		10	0.0031	0.0078	0.0008	0.0087	0.0009	0.0509
Contamination under DAVD								
			$\bar{X}_{RRSC}$		R_{RRSC}		S_{RRSC}	
b	k_3		*k-m=1,k=1*		*k-m=2,k=3*		*k-m=2,k=4*	
		n	*RRSC*	*USH*	*RRSC*	*USH*	*RRSC*	*USH*
1.13	0.80	2	0.0164	0.0107	0.0013	0.0003	0.0019	0.0005
		3	0.0114	0.0088	0.0014	0.0004	0.0016	0.0007
		4	0.0102	0.0086	0.0030	0.0006	0.0033	0.0013
		5	0.0081	0.0077	0.0039	0.0007	0.0043	0.0016
		7	0.0071	0.0072	0.0055	0.0013	0.0062	0.0023
		10	0.0056	0.0063	0.0081	0.0021	0.0081	0.0036
1.33	1.20	2	0.0238	0.0117	0.0010	0.0004	0.0016	0.0007
		3	0.0164	0.0106	0.0010	0.0006	0.0012	0.0011
		4	0.0135	0.0105	0.0022	0.0009	0.0021	0.0021
		5	0.0121	0.0094	0.0027	0.0012	0.0030	0.0025
		7	0.0089	0.0090	0.0042	0.0019	0.0042	0.0036
		10	0.0069	0.0080	0.0052	0.0036	0.0044	0.0060
1.69	2	2	0.0382	0.0146	0.0007	0.0006	0.0012	0.0010
		3	0.0269	0.0139	0.0007	0.0010	0.0012	0.0019
		4	0.0202	0.014	0.0010	0.0016	0.0015	0.0034
		5	0.0165	0.0136	0.0013	0.0021	0.0018	0.0044
		7	0.0122	0.0129	0.0019	0.0037	0.0025	0.0070
		10	0.0099	0.0111	0.0029	0.0066	0.0035	0.0117

Table 3.11, AADP (%) of different control charts with varying *k, m and*$\boldsymbol{k_3}$ at $\alpha = 0.0027$

$\bar{X}_{RRSC}$, $\bar{X}_{USH}$, $k-m=1$ and $k=1$								
n	k_3	**Chart Type**	**Burr(1)**	**Burr(2)**	**Gamma**	**Chi-square**	**Weibull**	**Lognormal**
5	**0.8**	***RRSC***	0.9	0.6	0.4	0.4	0.5	0.4
		USH	5.1	4.7	4.1	4.1	4.3	4.1
	1.6	***RRSC***	2.5	2	1.4	1.5	2	1.6
		USH	10.9	10.4	9.4	9.8	10.1	9.7
	2.8	***RRSC***	3.5	3.9	4.8	5.8	5.7	3.7
		USH	17.4	17.7	17.9	18.8	20.3	17.3
10	**0.8**	***RRSC***	0.5	0.5	0.2	0.2	0.3	0.4
		USH	2.8	2.8	2.4	2.4	2.6	1.9
	1.6	***RRSC***	1.4	1.1	0.5	0.9	0.9	0.8
		USH	5.8	5.4	4.4	5.2	5.2	5.1
	2.8	***RRSC***	1.5	1.5	1.3	2.3	1.9	2.8
		USH	9.1	8.9	8.6	9.6	10	10.4
R_{RRSC} , R_{USH} , $k-m=2$ and $k=3$								
n	k_3	**Chart Type**	**Burr(1)**	**Burr(2)**	**Gamma**	**Chi-square**	**Weibull**	**Lognormal**
5	**0.8**	***RRSC***	1.33	0.51	0.28	0.18	1.75	0.5
		USH	5.04	4.09	3.85	3.52	1.89	4.21
	1.6	***RRSC***	1.35	2.03	0.94	1.18	0.75	1.33
		USH	12.02	12.78	11.84	11.85	11.75	11.94
	2.8	***RRSC***	1.5	2.33	1.02	2.18	1.79	2.49
		USH	22.72	23.64	24.76	13.26	15.75	24.03
10	**0.8**	***RRSC***	1.69	1.05	0.23	0.45	1.56	0.34
		USH	4.9	4.37	3.02	2.94	1.71	3.76
	1.6	***RRSC***	1.92	1.17	0.32	0.36	0.52	0.94
		USH	11.51	10.59	9.42	9.43	9.3	10.68
	2.8	***RRBSC***	1.58	2.69	1.56	1.58	1.80	4.04
		USH	20.62	21.95	23.42	20.45	19.3	23.63
S_{RRSC} , S_{USH} , $k-m=2$ and $k=4$								
n	k_3	**Chart Type**	**Burr(1)**	**Burr(2)**	**Gamma**	**Chi-square**	**Weibull**	**Lognormal**
5	**0.8**	***RRSC***	2.56	1.95	0.24	0.7	0.89	0.99
		USH	6.19	5.58	3.99	4.21	2.68	4.51
	1.6	***RRSC***	4.67	4.31	2.63	2.63	2.66	3.44
		USH	14.87	14.54	13.18	13.18	13.06	13.54
	2.8	***RRSC***	6.93	5.52	8.81	8.81	10.66	8.36
		USH	26.14	25	30.86	30.86	31.55	28.13
10	**0.8**	***RRSC***	1.69	1.05	0.23	0.45	1.56	0.34
		USH	4.9	4.37	3.02	2.94	1.71	3.76
	1.6	***RRSC***	1.92	1.17	0.32	0.36	0.52	0.94
		USH	11.51	10.59	9.42	9.43	9.3	10.68
	2.8	***RRSC***	1.58	2.69	1.50	2.15	3.64	4.04
		USH	20.62	21.95	15.26	19.80	18.26	23.63

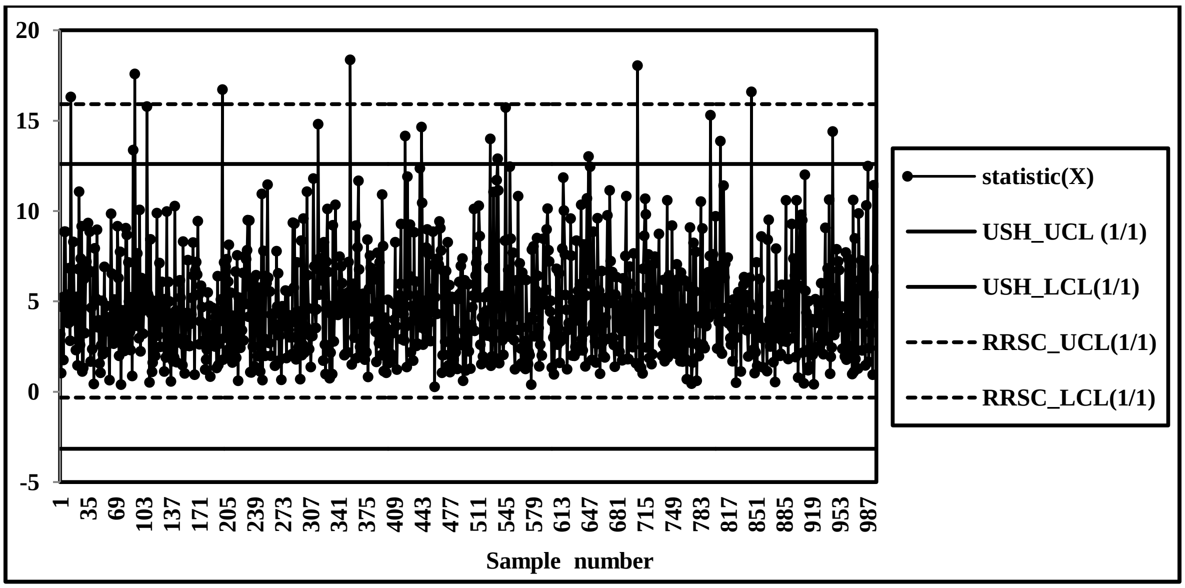
3.4 Numerical illustration

In order to provide numerical illustration of the charting structures of this study we have chosen a non-normal dataset (in-control) from Çiflikli (2006). The said dataset deals with a Chi-squared distribution having mean and standard deviation of 5 and 3.16 respectively. For the construction of runs rules based skewness correction control chart, we calculated skewness level (k_3 =1.26) and accordingly set the control limits by choosing appropriate coefficients for *m*=0 and *k*=1 at a prefixed value of α (for our purposes we set it at 1%). Finally, respective charting statistic is plotted against these limits. We have also included the limits for the same dataset using the usually used (normality based) *USH* limits. The resulting chart is presented in Figure 3.5(a) (for the sake of brevity we are presenting here the outcomes of $\bar{X}_{RRSC}$ *and* $\bar{X}_{USH}$ charts). Figure 3.5(a) shows that only 6 points (out of 1000) were observed outside the signaling limits by *RRSC* control chart whereas 17 points beyond the control limits are detected by *USH* (the same was reported by Çiflikli (2006) for *USH*. The expected numbers of signals are 10 based on our choice of α . We examine that the false alarm rate is inflated by *USH* which is not the case with *RRSC* (which is actually SC with 1/1 rule).

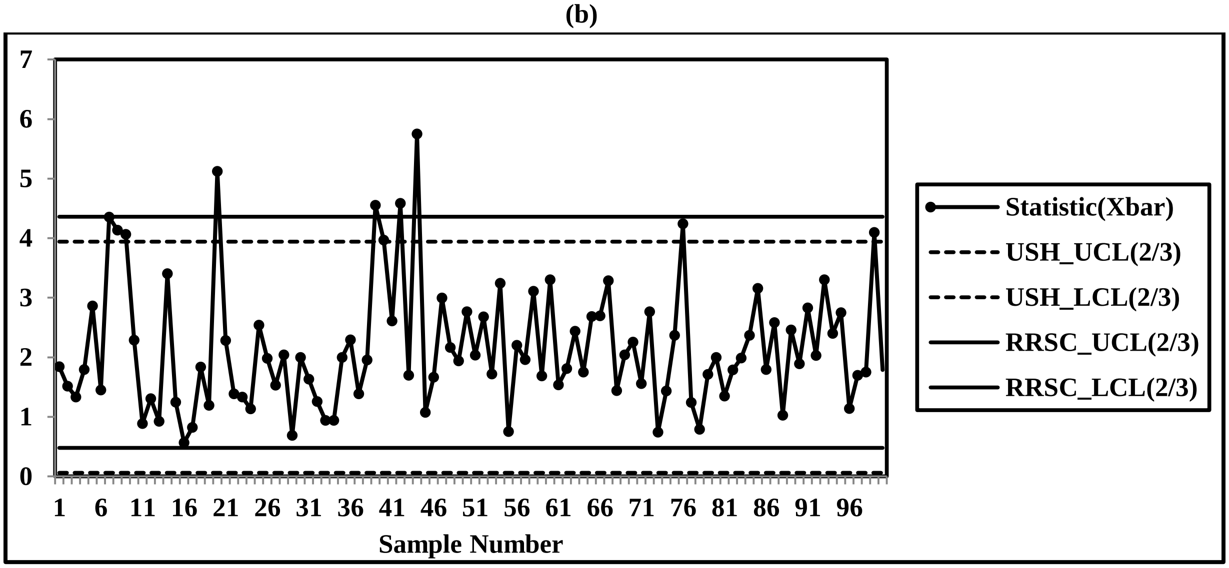
On the same guideline and motivation of Çiflikli (2006) we considered 30 in-control samples of size 5 having Chi-square distribution with mean=2 and standard deviation=2. We have estimated the control limits for *m*=1 and *k*=3 at $\alpha = 0.01$. Afterwards, 100 samples of size 5 are taken and tested using $\bar{X}_{RRSC}$ *and* $\bar{X}_{USH}$ control charts (see, Figure 3.5(b)). It is clearly indicated that only 1 signal is provided by the *RRSC* control chart whereas 7 signals (which are quite higher than the expected (1/100)) are given by the *USH* (with 2/3 rule). In addition, we have considered a shifted scenario of Chi-square (with a shift in mean of 0.5.) and constructed $\bar{X}_{RRSC}$ chart using 1/1 and 2/3 rules (cf. Figure 3.5(c)). Figure 3.5 (c) indicates that 3 signals detected by the 1/1 (at sample number 13, 14 and 15) and 10 signals detected by 2/3 (at sample number 12, 13, 14, 15, 16, 67, 68, 75, 76, 77). One can see the better detection ability of 2/3 over 1/1 by maintaining the false alarm rate.

Figure 3.5 numerical illustration using *RRSC* and *USH*

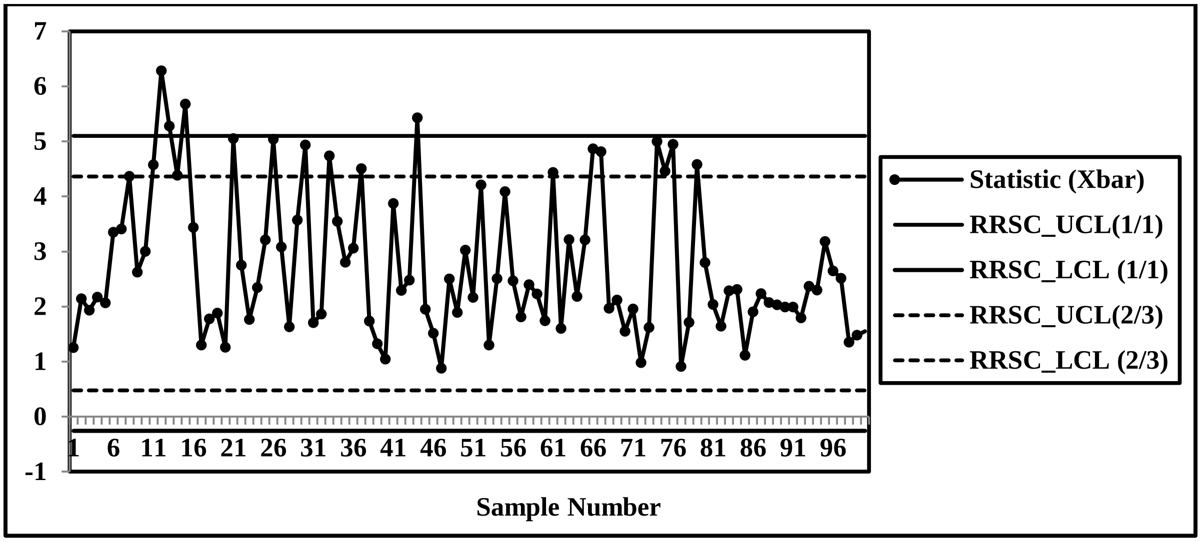
(a)



(b)



(c)


7
6
5
4
3
2
1
0
-1
1 6 11 16 21 26 31 36 41 46 51 56 61 66 71 76 81 86 91 96
Sample Number
Statistic (Xbar)
RRSC_UCL(1/1)
RRSC_LCL (1/1)
RRSC_UCL(2/3)
RRSC_LCL (2/3)

3.5 Summary and conclusions

In this chapter, we have investigated three commonly used $\bar{X}$, *R and S* control charts under three types of structures namely the usual Shewhart, skewness correction and runs based skewness correction. We have considered different probability distributions and contaminated normal cases as process models and used false alarm rate and probability to signals as criterion in order to compare their monitoring ability. We have observed that runs rules based skewness correction control charts have a good contribution towards maintaining the false alarm rate as well as a better detection ability in terms of probability to signal.

CHAPTER 4

ON THE EXTENDED USE OF AUXILIARY INFORMATION IN PROCESS MONITORING

In this chapter, we have extended the design structures of dual auxiliary information based on control charts for known process distribution, variety of sampling strategies and runs rules. We have also developed the design structures of dual auxiliary information based on control charts using the skewness correction (SC) method for unknown skewed distribution. The design structures with skewness correction method are based on the degree of skewness of the study variable, amount of correlation between study variable and auxiliary variable, and sample size. We have investigated the performance of the developed structures in term of probability to signals and false alarm rate by considering the skewed distribution, heavy tailed distribution and contamination environments. Outcomes of the current article showed that among known distributions based on control charts, extreme ranked set strategies based on control charts have higher probability of detecting an out-of-control signal and comparatively more robust than ranked set strategies based on control charts. Moreover skewness correction control charts under extreme ranked set strategies are more robust for small sample size, followed by ranked set strategies based on control charts for large sample size. Lastly, we also include a real life example for the monitoring of ground water variables.

4.1 Introduction

Statistical process control (SPC) consists of tools that are used to monitor a special cause of variation in the process parameters (location and dispersion), and most important one is quality control chart. A control chart is a graphical display for monitoring a process characteristic of interest and has wide applications in various disciplines such as in an industrial process, medical science and environmental science. The idea of quality control charts was initiated by Walter A. Shewhart in 1931. The most commonly used Shewhart-type control charts are $\bar{X}$, R, S, and S^2, whose design structures mainly depend on a study variable. Sometimes one may have extra (auxiliary) information, which is particularly or completely known and correlated with the study variable. For such situation, auxiliary information has been incorporated with the design structures of Shewhart-type control charts in the form of either ranking the units of interest or estimation of parameter and both (cf. Muttlak & Al-Sabah, 2003; Abujiya & Muttlak, 2004; Riaz, 2008a; Mehmood et al., 2013b; and Abbasi & Riaz, 2015).

One of the popular mechanisms of utilizing the auxiliary information is ranking the units of interest, which was given by Muttlak & Al-Sabah (2003). They developed the location control charts under single ranked set strategies, which include ranked set sampling (RSS), median ranked set sampling (MRSS), and extreme ranked set sampling (ERSS). Then, Abujiya & Muttlak (2004) extended the idea of single ranked set strategies to double ranked set strategies. They considered different variants of double ranked set strategies like single ranked set strategies and demonstrated that double ranked set strategies based on control charts are more efficient than single ranked set strategies based on control charts. In addition, control charts proposed by Muttlak & Al-Sabah

(2003), and Abujiya & Muttlak (2004) were based on one point decision rule, which usually considered less efficient for the detection of smaller shifts. The limitation of one point decision rule is discussed in many studies such as Montgomry (2009) and Riaz et al. (2011). Later on, Mehmood et al. (2013b) attached more runs rules with the usual ranked set strategies based structures and improved their detection ability towards smaller shifts.

Another popular mechanism for utilizing the auxiliary information has been seen at the estimation stage for designing a control chart. Riaz (2008a) considered the auxiliary information at estimation stage and proposed a control chart for the monitoring of location parameter. Furthermore, some studies on the similar lines are found in the literature of statistical process control like Riaz et al. (2014) and Ahmad et al. (2014).

Yu & Lam (1997) and Muttlak (2001) recommended the use of auxiliary information for dual purposes instead of using it either for ranking of units or estimation. For the said purpose, they suggested regression type estimators under ranked set sampling (cf. Yu & Lam, 1997), median ranked set sampling (cf. Muttlak, 2001), and extreme ranked set sampling (cf. Muttlak, 2001). Moreover, one recent application of dual use of auxiliary information can be seen for designing the structures of control charts. Abbasi & Riaz (2015) promoted the idea of dual use of auxiliary information in control charts. They proposed location control charts based on known distribution in which they considered normal distribution, single ranked set strategies (include RSS, ERSS and MRSS) and one point decision rule. It is important to mention here control charts under the known normal distribution are not always remain beneficial for many practical processes, especially when process distribution is other than normal or process distribution is unknown.

In this chapter, we mainly intend to propose more generalized structures of dual auxiliary information based on control charts for known process distributions (following the Abbasi & Riaz, 2015) as well as for unknown skewed processes. For designing the known process distribution based on control charts, we consider variety of distributions, sampling strategies and runs rules. In more details, process distributions include bivariate normal (BN), bivariate t (BT) and bivariate lognormal (BLN), whereas sampling strategies include ranked set sampling (RSS), extreme ranked set sampling (ERSS), double ranked set sampling (DRSS), and double extreme ranked set sampling (DERSS). Furthermore, we use skewness correction method following the Chan & Cui (2003) in order to design the control charts for unknown skewed processes under variety of sampling strategies.

The organization of rest of the chapter is as follows: In Section 4.2, we explain regression estimator under different sampling strategies. In Section 4.3, we propose a set of location control charts for known as well as for unknown process distributions separately. In Section 4.4, we investigate the performance of proposed charts in term of probability to signals (also termed as power) and false alarm rate. Section 4.5 elaborate results and discussion based on the performance measures. Section 4.6 presents an application of the proposed charts in groundwater monitoring. Section 4.7 covers the concluding remarks of the whole study.

4.2 Regression estimators under different sampling strategies

In the subsequent sections, we illustrate regression estimators under different ranked set strategies, which utilize the auxiliary information for ranking the units as well as for estimation of parameters (dual propose). Firstly, we explain regression estimators under single ranked set strategies, which were introduced by Yu & Lam (1997), and Muttlak (2001). Then, on the same guideline of Yu & Lam (1997) and Muttlak (2001) we propose regression estimators under double ranked set strategies.

4.2.1 Regression estimator under single ranked set strategies

Let $\left(Y_{[i,rss],j}, X_{(i,rss),j}\right)$ and $\left(Y_{[i,erss],j}, X_{(i,erss),j}\right)$, where $i = 1,2,3,\dots,n$ and $j = 1,2,3,\dots,r$ denotes jth random sample of size n collected under ranked set sampling (RSS) and extreme ranked set sampling (ERSS) respectively. The details of RSS and ERSS can be seen in the articles of Muttlak & Al-Sabah (2003), Abujiya & Muttlak (2004), Mehmood et al. (2013b), and Abbasi & Riaz (2015). Then regression estimators (see Yu & Lam, 1997; and Muttlak, 2001) of **population mean** for the jth random sample of size n are given below:

$$V_{RSS,j} = \bar{Y}_{RSS,j} + \left(r_{YX_{RSS,j}} \frac{s_{Y_{RSS,j}}}{s_{X_{RSS,j}}}\right)\left[\mu_X - \bar{X}_{RSS,j}\right], \qquad j = 1,2,3,\dots,r,$$

$$V_{ERSS,j} = \bar{Y}_{ERSS,j} + \left(r_{YX_{ERSS,j}} \frac{s_{Y_{ERSS,j}}}{s_{X_{ERSS,j}}}\right)\left[\mu_X - \bar{X}_{ERSS,j}\right], \quad j = 1,2,3,\dots,r.$$

Variances of the above estimators are given as follow: $\sigma^2_{V_{RSS,j}} = \frac{\sigma_Y^2}{n}(1-\rho_{YX}^2)\left[1 + E\left(\frac{\bar{Z}^2_{RSS}}{S^2_{Z_{RSS}}}\right)\right]$, $\sigma^2_{V_{ERSS,j}} = \frac{\sigma_Y^2}{n}(1-\rho_{YX}^2)\left[1 + E\left(\frac{\bar{Z}^2_{ERSS}}{S^2_{Z_{ERSS}}}\right)\right]$, where $\bar{Y}_{RSS,j}$, $\bar{Y}_{ERSS,j}$ $\bar{X}_{RSS,j}$, and $\bar{X}_{ERSS,j}$ are sample means for the jth RSS and ERSS sample of size n. Moreover, other quantities are defined as: $\bar{Z}_{RSS} = \frac{1}{n}\sum_{i=1}^{n} Z_{(i,rss),j}$, $S^2_{Z_{RSS}} = \frac{1}{n}\sum_{i=1}^{n}\left[Z_{(i,rss),j} - \bar{Z}_{RSS}\right]$, $Z_{(i,rss),j} = \frac{X_{(i,rss),j}-\mu_X}{\sigma_X}$, $\bar{Z}_{ERSS} = \frac{1}{n}\sum_{i=1}^{n} Z_{(i,erss),j}$, $S^2_{Z_{ERSS}} = \frac{1}{n}\sum_{i=1}^{n}\left[Z_{(i,erss),j} - \bar{Z}_{ERSS}\right]$, $Z_{(i,erss),j} = \frac{X_{(i,erss),j}-\mu_X}{\sigma_X}$.

4.2.2 Regression estimator under double ranked set strategies

On the same guideline of Yu & Lam (1997), and Muttlak (2001) we propose regression estimators under double ranked set strategies, which include double ranked set sampling (DRSS) and double extreme ranked set sampling (DERSS). Let $\left(Y_{[i,drss],j}, X_{(i,drss),j}\right)$, and $\left(Y_{[i,derss],j}, X_{(i,derss),j}\right)$, where $i = 1,2,3,\ldots,n$ and $j = 1,2,3,\ldots,r$, denote jth random sample of size n collected by using the procedure of DRSS, and DERSS respectively. The details of double ranked set strategies have given in various studies such as Abujiya & Muttlak (2004), and Mehmood et al. (2013b). Then, regression estimators of the **population mean** for the jth random sample of size n are as follow:

$$V_{DRSS,j} = \bar{Y}_{DRSS,j} + \left(r_{YX_{DRSS,j}} \frac{s_{Y_{DRSS,j}}}{s_{X_{DRSS,j}}}\right)\left[\mu_X - \bar{X}_{DRSS,j}\right], \qquad j = 1,2,3,\ldots,r,$$

$$V_{DERSS,j} = \bar{Y}_{DERSS,j} + \left(r_{YX_{DERSS,j}} \frac{s_{Y_{DERSS,j}}}{s_{X_{DERSS,j}}}\right)\left[\mu_X - \bar{X}_{DERSS,j}\right], \quad j = 1,2,3,\ldots,r.$$

Variances of the above estimators are given as: $\sigma^2_{V_{DRSS,j}} = \frac{\sigma_Y^2}{n}(1-\rho_{YX}^2)\left[1 + E\left(\frac{\bar{Z}^2_{DRSS}}{S^2_{Z_{DRSS}}}\right)\right]$,

, $\sigma^2_{V_{DERSS,j}} = \frac{\sigma_Y^2}{n}(1-\rho_{YX}^2)\left[1 + E\left(\frac{\bar{Z}^2_{DERSS}}{S^2_{Z_{DERSS}}}\right)\right]$, where where $\bar{Y}_{DRSS,j}$, $\bar{Y}_{DERSS,j}$ $\bar{X}_{DRSS,j}$, and $\bar{X}_{DERSS,j}$ are sample means for the jth DRSS and DERSS sample of size n , like in single ranked set strategies. Moreover, other quantities are defined as: $\bar{Z}_{DRSS} = \frac{1}{n}\sum_{i=1}^{n} Z_{(i,drss),j}$, $S^2_{Z_{DRSS}} = \frac{1}{n}\sum_{i=1}^{n}\left[Z_{(i,drss),j} - \bar{Z}_{DRSS}\right]$, $Z_{(i,drss),j} = \frac{X_{(i,drss),j} - \mu_X}{\sigma_X}$, $\bar{Z}_{DERSS} = \frac{1}{n}\sum_{i=1}^{n} Z_{(i,drss),j}$, $S^2_{Z_{DERSS}} = \frac{1}{n}\sum_{i=1}^{n}\left[Z_{(i,derss),j} - \bar{Z}_{DERSS}\right]$, $Z_{(i,derss),j} = \frac{X_{(i,derss),j} - \mu_X}{\sigma_X}$.

4.3 Proposed location control charts

In this section we develop two design structures of dual auxiliary information based control charts using the regression estimators (see, section 4.2). In the following section control charts for known distributions and for unknown process distribution (skewed and heavy tailed) are denoted by $V_{(G,D)}$ and $V_{(G,SC)}$ respectively, where subscript G denotes a sampling strategy, D refers to a bivariate distribution and SC represents skewness correction method.

4.3.1 Design structure when process distribution is known

Let $V_{G,1}, V_{G,2,}, \ldots, V_{G,r}$, denote r plotting statistics based on the samples of size n that are collected using any of the sampling strategy G (RSS, ERSS, DRSS, and DERSS) under consideration with known bivariate distribution D (such as BN, BT and BLN) of the process characteristics. The design structure of the proposed control charts following the Abbasi & Riaz (2015) are:

$$LCL_{V_{(G,D)}} = \mu_Y + L_{(G,D,n,\rho_{YX},\frac{\alpha}{2})}\sigma_Y, \text{ and } UCL_{V_{(G,D)}} = \mu_Y + L_{(G,D,n,\rho_{YX},1-\frac{\alpha}{2})}\sigma_Y, \qquad (4.1)$$

where μ_Y and σ_Y are population mean and standard deviation of the variable of interest Y, $L_{(G,D,n,\rho_{YX},\frac{p}{2})}$ and $L_{(G,D,n,\rho_{YX},1-\frac{p}{2})}$ are control limit factors depend on sampling strategy G , distribution D, sample size n and correlation ρ_{YX} between Y and X. In order to make the design structure (given in equation 4.1) more sensitive for smaller and moderate shifts, we have defined and attached more runs rules (cf. Riaz et al. 2011; and Mehmood et al., 2013b & 2014): "Process can be declared out-of-control, if at least $k - m$ values of the test statistic ($V_{G,j}, j = 1,2,3..r$) out of k consecutive values of test statistic either exceed the lower control limit $LCL_{V_{(G,D)}}$ or upper control limit $UCL_{V_{(G,D)}}$".The following statement can be represented as $k-m|k$ with the condition that $0 \leq m \leq k-1$.

$$LCL_{V_{(G,D)}} = \mu_Y + L_{(G,D,n,k-m,k,\rho_{YX},\frac{p}{2})}\sigma_Y,$$

$$UCL_{V_{(G,D)}} = \mu_Y + L_{(G,D,n,k-m,k,\rho_{YX},1-\frac{p}{2})}\sigma_Y, \tag{4.2}$$

where $k-m$ denotes the decision observations in order to declare a process out-of-control and k is the total observations in a given rule. Rests of the quantities are defined earlier. It is important to mention here, $V_{(RSS,BN)}$ and $V_{(ERSS,BN)}$ control charts are the special cases of existing dual auxiliary information based on control charts when $k-m=1, k=1$.

Furthermore, control limits factors are derived through Monte Carlo simulation. Before illustrating the computation procedure, it is important to mention here for bivariate lognormal distribution (which is a case of skewed distribution) we consider $Y = lnQ$ and $Y = lnW$ with means and variances are $\mu_Y = ln(\mu_Q) - \left(\frac{\sigma_Q^2}{2}\right)$, $\mu_X = l\,n(\mu_W) - \left(\frac{\sigma_W^2}{2}\right)$, $\sigma_Y^2 = ln\left(1 + \frac{\sigma_Q^2}{\mu_Q^2}\right)$ and $\sigma_X^2 = ln\left(1 + \frac{\sigma_W^2}{\mu_W^2}\right)$, where (Q, W) follows bivariate lognormal distribution with means are μ_Q and μ_W , and variances are σ_Q^2 and σ_W^2 respectively (for some detail see Appendix B). Details of the following transformation are provided by Stedinger et al. (1993) and, Yerel & Konuk (2009). Moreover, it is helpful to bring the non-linear random variable into linear form. Assumption of linearity of a variable is usually considered important for using a difference estimator and regression estimator. Yu & Lam (1997), and Muttlak (2001) concluded that regression estimators works efficiently when (Y, X) is linear. They also concluded some kind of transformation can be used for using the regression estimator when variable under consideration are non-linear.

Afterwards, we consider different value of design parameters $(G, D\ k-m, k, n, \rho_{YX})$, random samples are generated from a given bivariate distribution D ($\mu_Y = 0$, $\mu_X = 0$, $\sigma_Y^2 = 1$, $\sigma_X^2 = 1$, and $v = 30$) and the charting statistic are calculated 10^5 times. Finally, the control limits factors (L) are obtained by taking the $\left(1-\frac{p}{2}\right)$th and $\left(\frac{p}{2}\right)$th quantiles of the sampling distribution of test statistic for a given value of $G, D, k-m, k, n,$ and ρ_{YX}. Moreover, in this study we have tabulated control limits factors for selective choices of control charts in the form of Tables 4.1-4.3 under varying values of design parameters $(G, D, k-m, k, n, \rho_{YX})$ at $\alpha = 0.0027$. Similarly, control limits factors can be derived for the other choice of design parameters and α.

4.3.2 Design structure when process distribution is unknown

Let $V_{G,1}, V_{G,2,}, \ldots, V_{G,r}$, denote r plotting statistics based on the samples of size n that are collected using any of the sampling strategy G (include RSS, ERSS, DRSS, and DERSS) under consideration with known value of skewness of the study variable Y (k_3), means (μ_Y and μ_X), standard deviations (σ_Y and σ_X), and correlation (ρ_{YX}) between the study variable Y and auxiliary variable X. The proposed skewness correction control charts for unknown skewed distribution following the Chan & Cui (2003) are:

$$LCL_{V_{(G,SC)}} = \mu_Y + \left[Z_{\left(\frac{\alpha}{2}\right)}k_2 + c_4^*\right]\sigma_Y, \text{ and } UCL_{V_{(G,SC)}} = \mu_Y + \left[Z_{\left(1-\frac{\alpha}{2}\right)}k_2 + c_4^*\right]\sigma_Y, \quad (4.3)$$

where $c_4^* = \frac{\frac{\left(Z^2_{\left(\frac{\alpha}{2}\right)}-1\right)k_3(V)k_2}{6}}{1+0.2k_3(V)} = \frac{\frac{\left(Z^2_{\left(1-\frac{\alpha}{2}\right)}-1\right)k_3(V)k_2}{6}}{1+0.2k_3(V)}$ is skewness adjustment factor, $Z_{\left(\frac{\alpha}{2}\right)}$ and $Z_{\left(1-\frac{\alpha}{2}\right)}$ are $\left(\frac{\alpha}{2}\right)$th and $\left(1-\frac{\alpha}{2}\right)$th quantiles of the standard normal distribution for a given false alarm rate α, $k_3(V)$ refers to skewness of the statistics ($V_{G,j}$ *,j=1,2,…,r)* and k_2 is the ratio of standard deviation of statistics $V_{G,j}$ and standard deviation of the study variableσ_Y i.e. $k_2 = \frac{\sigma_{V_{G,j}}}{\sigma_Y} = \sqrt{\frac{1}{n}(1-\rho_{YX}^2)\left[1+E\left(\frac{\bar{Z}_G^2}{S_{Z_G}^2}\right)\right]}$. We have computed k_2 through Monte Carlo simulation in the following ways: For a given value of n, ρ_{YX} and G, random samples are generated (10^5 times) from any of the bivariate distribution with no restriction on the choice of parameters, computed the statistics ($V_{G,j}$, *j=1,2,…* 10^5), and finally, ratio of the standard deviation of computed statistic and standard deviation of the study variable σ_Y is our desired k_2. We have provided k_2 for some selective choices n and ρ_{YX} with varying sampling strategy given in Table 4.4.

An alternative approach of representing the Equation 4.3 is as follows:

$$LCL_{V_{(G,SC)}} = \mu_Y + B_L^*\sigma_Y, \text{ and } UCL_{V_{(G,SC)}} = \mu_Y + B_U^*\sigma_Y, \tag{4.4}$$

where $B_L^* = \left[Z_{\left(\frac{\alpha}{2}\right)}k_2 + c_4^*\right]$ and $B_U^* = \left[Z_{\left(1-\frac{\alpha}{2}\right)}k_2 + c_4^*\right]$ are control limits factors.

After defining the control charting structures (see Equations 4.3-4.4) for unknown skewed distributions, we have calculated the skewness adjustment factor c_4^* following the Chan & Cui (2003) for different choices of skewness level (depends on value of parameters), sample size n, correlation ρ_{YX} and sampling strategy G at $\alpha = 0.0027$. The results of the following constant are tabulated in Table 4.5. We also provided the control limits coefficients (B_L^*, B_U^*) in Table 4.6.

Table 4.1. Control limits factors of $\boldsymbol{V}_{(\boldsymbol{ERSS},\boldsymbol{BN})}$ control charts with different choices of $\boldsymbol{k-m},\boldsymbol{k},\boldsymbol{n}$ **and** $\boldsymbol{\rho_{YX}}$at $=\mathbf{0.0027}$.

$\boldsymbol{\rho_{YX}}$	n	**Control limits factors**	$\boldsymbol{k-m\|k}$					
			1|1	**2|3**	**2|4**	**9|9**	**8|9**	**7|9**
0.30	**4**	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.57	-1.1022	-1.1719	-0.3233	-0.4383	-0.5441
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.5664	1.0933	1.1627	0.3239	0.4374	0.5418
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.4365	-1.0023	-1.0665	-0.2918	-0.3959	-0.4908
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.4355	1.0036	1.069	0.297	0.4014	0.4972
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.0757	-0.7554	-0.8042	-0.2242	-0.3034	-0.3764
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.0901	0.763	0.8125	0.2236	0.3042	0.3773
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.7189	-0.5018	-0.5355	-0.1489	-0.201	-0.2495
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.712	0.5038	0.5356	0.1474	0.2	0.2477
0.50	**4**	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.3395	-0.9594	-1.0185	-0.285	-0.3854	-0.4775
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.3407	0.9575	1.015	0.2851	0.3859	0.4767
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.2165	-0.8704	-0.9259	-0.2585	-0.3496	-0.434
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.2195	0.8732	0.9253	0.2567	0.3483	0.4331
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.9334	-0.6672	-0.7094	-0.1971	-0.2675	-0.3313
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.9266	0.6635	0.7044	0.1984	0.268	0.332
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.6168	-0.4399	-0.4681	-0.1307	-0.1769	-0.2193
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.6091	0.4402	0.4683	0.1305	0.1769	0.2189
0.75	**4**	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.1926	-0.8613	-0.9116	-0.2553	-0.3452	-0.428
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.2093	0.8598	0.9129	0.2586	0.3498	0.433
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.0781	-0.7783	-0.8274	-0.2304	-0.313	-0.3882
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.0814	0.7776	0.8271	0.2338	0.3158	0.3909
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.8304	-0.5934	-0.6315	-0.1775	-0.2394	-0.2967
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.8326	0.5966	0.6332	0.1763	0.2397	0.2977
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.5412	-0.3932	-0.4172	-0.1168	-0.1575	-0.1955
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.5477	0.3918	0.4165	0.1171	0.1584	0.1963
0.90	**4**	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.0184	-0.7336	-0.7781	-0.2198	-0.2965	-0.3661
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.02	0.7366	0.7826	0.2211	0.2975	0.3682
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.9307	-0.6683	-0.7108	-0.1996	-0.2697	-0.3344
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.9255	0.6715	0.7157	0.2003	0.2709	0.3358
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.7053	-0.5122	-0.5449	-0.1518	-0.2056	-0.255
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.7092	0.5106	0.5418	0.152	0.2059	0.2562
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.4639	-0.3364	-0.3564	-0.1011	-0.1368	-0.1694
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.4664	0.3378	0.3587	0.1008	0.1361	0.168

Table 4.2. Control limits factors of $V_{(ERSS,BT)}$ control charts with different choices of $\boldsymbol{k-m,k,n}$ **and** $\boldsymbol{\rho_{YX}}$at $= \mathbf{0.0027}$.

ρ_{YX}	n	Control limits factors	$k-m\|k$					
			1|1	2|3	2|4	9|9	8|9	7|9
0.30	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.1022	-1.1719	-0.3233	-0.4383	-0.5441	-1.1022
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.0933	1.1627	0.3239	0.4374	0.5418	1.0933
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.0023	-1.0665	-0.2918	-0.3959	-0.4908	-1.0023
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.0036	1.069	0.297	0.4014	0.4972	1.0036
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.7554	-0.8042	-0.2242	-0.3034	-0.3764	-0.7554
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.763	0.8125	0.2236	0.3042	0.3773	0.763
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.5018	-0.5355	-0.1489	-0.201	-0.2495	-0.5018
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.5038	0.5356	0.1474	0.2	0.2477	0.5038
0.50	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.9594	-1.0185	-0.285	-0.3854	-0.4775	-0.9594
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.9575	1.015	0.2851	0.3859	0.4767	0.9575
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.8704	-0.9259	-0.2585	-0.3496	-0.434	-0.8704
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.8732	0.9253	0.2567	0.3483	0.4331	0.8732
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.6672	-0.7094	-0.1971	-0.2675	-0.3313	-0.6672
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.6635	0.7044	0.1984	0.268	0.332	0.6635
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.4399	-0.4681	-0.1307	-0.1769	-0.2193	-0.4399
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.4402	0.4683	0.1305	0.1769	0.2189	0.4402
0.75	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.8613	-0.9116	-0.2553	-0.3452	-0.428	-0.8613
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.8598	0.9129	0.2586	0.3498	0.433	0.8598
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.7783	-0.8274	-0.2304	-0.313	-0.3882	-0.7783
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.7776	0.8271	0.2338	0.3158	0.3909	0.7776
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.5934	-0.6315	-0.1775	-0.2394	-0.2967	-0.5934
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.5966	0.6332	0.1763	0.2397	0.2977	0.5966
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.3932	-0.4172	-0.1168	-0.1575	-0.1955	-0.3932
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.3918	0.4165	0.1171	0.1584	0.1963	0.3918
0.90	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.7336	-0.7781	-0.2198	-0.2965	-0.3661	-0.7336
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.7366	0.7826	0.2211	0.2975	0.3682	0.7366
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.6683	-0.7108	-0.1996	-0.2697	-0.3344	-0.6683
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.6715	0.7157	0.2003	0.2709	0.3358	0.6715
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.5122	-0.5449	-0.1518	-0.2056	-0.255	-0.5122
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.5106	0.5418	0.152	0.2059	0.2562	0.5106
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.3364	-0.3564	-0.1011	-0.1368	-0.1694	-0.3364
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.3378	0.3587	0.1008	0.1361	0.168	0.3378

Table 4.3. Control limits factors of $\boldsymbol{V}_{(\boldsymbol{ERSS},\boldsymbol{BLN})}$ control charts with different choices of $\boldsymbol{k}-\boldsymbol{m},\boldsymbol{k},\boldsymbol{n}$ **and** $\boldsymbol{\rho}_{\boldsymbol{YX}}$at $=\mathbf{0.0027}$.

$\boldsymbol{\rho}_{\boldsymbol{YX}}$	n	Control limits factors	$\boldsymbol{k}-\boldsymbol{m}\vert\boldsymbol{k}$					
			1|1	2|3	2|4	9|9	8|9	7|9
0.30	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.5518	-1.1044	-1.1724	-0.3235	-0.4381	-0.5439
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.5516	1.095	1.1653	0.3234	0.4396	0.5429
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.4237	-0.992	-1.0561	-0.2928	-0.398	-0.4943
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.4156	1.0009	1.065	0.2916	0.3953	0.4901
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.1035	-0.7634	-0.8109	-0.226	-0.3044	-0.3774
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.0775	0.759	0.8089	0.2238	0.3026	0.3759
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.717	-0.5037	-0.5364	-0.1467	-0.1989	-0.2476
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.7119	0.504	0.5356	0.1488	0.2013	0.2493
0.50	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.3319	-0.9637	-1.0254	-0.2883	-0.3883	-0.4791
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.3337	0.9571	1.0124	0.2819	0.3833	0.4777
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.2182	-0.8745	-0.9269	-0.2592	-0.3509	-0.4347
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.2171	0.8669	0.92	0.2577	0.3489	0.4314
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.929	-0.6676	-0.7086	-0.1971	-0.2673	-0.3321
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.9342	0.668	0.7122	0.1971	0.2672	0.3313
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.6082	-0.44	-0.4661	-0.1296	-0.1761	-0.2185
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.6035	0.4369	0.4628	0.1301	0.1763	0.2184
0.75	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.1891	-0.8609	-0.9144	-0.2555	-0.3465	-0.4299
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.1964	0.8597	0.9144	0.256	0.3476	0.4314
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.0959	-0.7832	-0.8337	-0.2326	-0.3148	-0.3907
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.0719	0.7828	0.8297	0.2313	0.3148	0.3899
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.83	-0.596	-0.6319	-0.1777	-0.2402	-0.2982
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.8264	0.5978	0.6359	0.1779	0.2408	0.2978
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.5418	-0.394	-0.4192	-0.117	-0.1585	-0.1961
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.5457	0.3949	0.4174	0.1168	0.1579	0.1955
0.90	4	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-1.0318	-0.7399	-0.7912	-0.2208	-0.298	-0.3683
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.0163	0.7368	0.7809	0.2194	0.2968	0.3676
	5	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.9311	-0.6684	-0.7089	-0.2003	-0.2703	-0.3353
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.9271	0.6679	0.7093	0.1996	0.2707	0.3354
	6	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.7129	-0.5119	-0.5443	-0.1529	-0.2073	-0.2565
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.7132	0.5122	0.5433	0.1521	0.2057	0.2547
	8	$L_{(G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	-0.4658	-0.337	-0.3574	-0.1008	-0.1364	-0.1692
		$L_{(G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	0.4684	0.3367	0.3582	0.1	0.1356	0.1683

Table 4.4. $\boldsymbol{K_2}$ with varying value of $\boldsymbol{n}$, $\boldsymbol{\rho_{YX}}$ and $\boldsymbol{G}$.

$\boldsymbol{n}$	$\boldsymbol{\rho_{YX}}$	$\boldsymbol{G}$			
		RSS	ERSS	DRSS	DERSS
5	0.50	0.41	0.4	0.42	0.39
	0.75	0.31	0.31	0.31	0.3
	0.90	0.21	0.2	0.2	0.2
7	0.50	0.34	0.33	0.33	0.33
	0.75	0.26	0.25	0.25	0.25
	0.90	0.17	0.17	0.17	0.17
9	0.50	0.29	0.29	0.29	0.28
	0.75	0.23	0.22	0.22	0.22
	0.90	0.15	0.15	0.15	0.15

Table 4.5. Skewness correction factor $\boldsymbol{c_4^*}$ of $\boldsymbol{V_{(G,SC)}}$ control charts with varying values of $\boldsymbol{n}$, $\boldsymbol{\rho_{YX}}$ $\boldsymbol{k_3}$ and $\boldsymbol{G}$ at $\boldsymbol{\alpha = 0.0027}$

$\boldsymbol{n}$	$\boldsymbol{\rho_{YX}}$	$\boldsymbol{k_3}$	$\boldsymbol{G}$			
			RSS	ERSS	DRSS	DERSS
5	0.50	0.8	0.18	0.18	0.17	0.16
		2	0.46	0.44	0.42	0.35
		2.4	0.59	0.48	0.52	0.4
	0.75	0.8	0.11	0.12	0.09	0.1
		2	0.31	0.25	0.26	0.21
		2.4	0.32	0.27	0.3	0.24
	0.90	0.8	0.05	0.05	0.05	0.04
		2	0.08	0.1	0.11	0.09
		2.4	0.16	0.15	0.15	0.11
7	0.50	0.8	0.13	0.1	0.13	0.11
		2	0.33	0.28	0.31	0.24
		2.4	0.38	0.31	0.34	0.29
	0.75	0.8	0.07	0.06	0.07	0.06
		2	0.18	0.17	0.18	0.15
		2.4	0.23	0.18	0.23	0.16
	0.90	0.8	0.03	0.03	0.03	0.02
		2	0.08	0.06	0.08	0.07
		2.4	0.11	0.08	0.1	0.08
9	0.50	0.8	0.02	0.09	0.08	0.08
		2	0.22	0.19	0.22	0.18
		2.4	0.28	0.23	0.27	0.27
	0.75	0.8	0.06	0.05	0.06	0.05
		2	0.15	0.12	0.12	0.12
		2.4	0.16	0.13	0.15	0.14
	0.90	0.8	0.02	0.03	0.02	0.02
		2	0.06	0.05	0.06	0.05
		2.4	0.08	0.06	0.08	0.06

Table 4.6. control limits coefficients ($\boldsymbol{B_L^*}$, $\boldsymbol{B_U^*}$) of $\boldsymbol{V_{(G,SC)}}$ control charts with varying values of $\boldsymbol{n}$, $\boldsymbol{\rho_{YX}}$ $\boldsymbol{k_3}$ and $\boldsymbol{G}$ at $\boldsymbol{\alpha = 0.0027}$.

$\boldsymbol{n}$	$\boldsymbol{\rho_{YX}}$	$\boldsymbol{k_3}$	$\boldsymbol{G}$							
			RSS		ERSS		DRSS		DERSS	
			$\boldsymbol{B_L^*}$	$\boldsymbol{B_U^*}$	$\boldsymbol{B_L^*}$	$\boldsymbol{B_U^*}$	$\boldsymbol{B_L^*}$	$\boldsymbol{B_U^*}$	$\boldsymbol{B_L^*}$	$\boldsymbol{B_U^*}$
5	0.50	0.8	1.04	1.40	1.02	1.37	1.1	1.44	1	1.33
		2	0.76	1.68	0.75	1.64	0.85	1.68	0.81	1.51
		2.4	0.63	1.81	0.71	1.68	0.74	1.79	0.76	1.56
	0.75	0.8	0.83	1.04	0.82	1.05	0.83	1.02	0.78	0.99
		2	0.63	1.25	0.69	1.19	0.67	1.18	0.68	1.09
		2.4	0.61	1.26	0.67	1.21	0.62	1.22	0.65	1.13
	0.90	0.8	0.57	0.68	0.56	0.66	0.56	0.66	0.56	0.10
		2	0.54	0.7	0.51	0.71	0.49	0.72	0.51	0.15
		2.4	0.46	0.78	0.47	0.76	0.46	0.75	0.46	0.17
7	0.5	0.8	0.89	1.14	0.89	1.09	0.87	1.13	0.87	1.09
		2	0.69	1.35	0.71	1.28	0.69	1.31	0.74	1.22
		2.4	0.63	1.4	0.68	1.31	0.65	1.34	0.69	1.27
	0.75	0.8	0.71	0.85	0.69	0.82	0.69	0.83	0.68	0.81
		2	0.6	0.96	0.59	0.92	0.58	0.95	0.6	0.89
		2.4	0.55	1.01	0.58	0.93	0.54	0.99	0.59	0.9
	0.90	0.8	0.49	0.54	0.48	0.53	0.47	0.54	0.47	0.52
		2	0.43	0.59	0.44	0.57	0.43	0.58	0.43	0.57
		2.4	0.4	0.62	0.42	0.59	0.41	0.6	0.42	0.58
9	0.50	0.8	0.86	0.91	0.78	0.95	0.78	0.95	0.76	0.93
		2	0.66	1.1	0.67	1.06	0.64	1.08	0.67	1.02
		2.4	0.6	1.17	0.63	1.1	0.59	1.13	0.57	1.12
	0.75	0.8	0.62	0.74	0.62	0.73	0.6	0.71	0.62	0.71
		2	0.53	0.83	0.55	0.79	0.53	0.78	0.55	0.78
		2.4	0.52	0.85	0.54	0.8	0.51	0.8	0.53	0.8
	0.90	0.8	0.42	0.47	0.42	0.47	0.42	0.47	0.42	0.46
		2	0.38	0.5	0.39	0.5	0.39	0.5	0.4	0.49
		2.4	0.36	0.52	0.39	0.5	0.37	0.52	0.39	0.5

4.4 Performance evaluation of proposed control charts

In section 4.3 we have developed two control charting structures of $V_{(G,D)}$ and $V_{(G,SC)}$ control charts, which are given in Equations 4.2 & 4.4. In the following section we investigate the performance of $V_{(G,D)}$ and $V_{(G,SC)}$ control charts. For the said purposes, we use probability to signals and false alarm rate as the performance measures for evaluating the performance of $V_{(G,D)}$ and $V_{(G,SC)}$ control charts respectively. Probability to signals is the detection probability of a given control chart when the process actually out-of-control. The false alarm rate is the probability of process breaching outside the control limits ($LCL_{V_{(G,SC)}}, UCL_{V_{(G,SC)}}$) when it is in fact in-control.

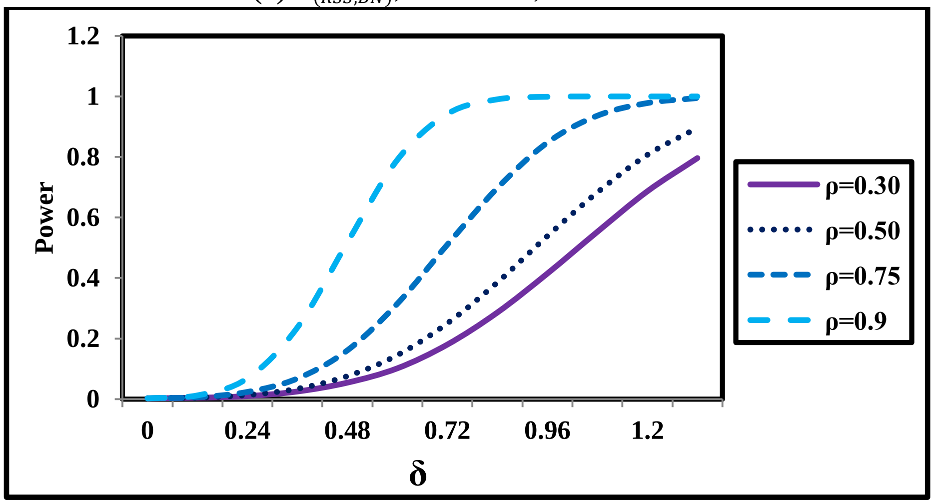
4.4.1 Performance evaluation of the $V_{(G,D)}$ control charts

In order to compute the probability to signals, we assume shift δ occurs in the process parameter of the variable of interest μ_Y. The shift in the process parameter of the variable of interest Y is defined as: $\mu_{Y(\delta)} = \mu_Y + \delta\sigma_Y$. Moreover, $\delta = 0$ implies that no shift occurs in the process parameter of the variable of interest and the process behaving naturally, whereas, $\delta > 0$ means that a special cause of variation is interrupting the process and switching it from in-control state to out-of-control. The shift can occur in the variable of interest while dealing the auxiliary information based on location control charts is illustrated by Riaz (2008a), Riaz et al. (2014), and Ahmad et al. (2014). In more details, for the computation of power, we assume process follows bivariate distribution D (such as BN , BT and BLN) with known in-control parameters are $\mu_Y = 0$, $\mu_X = 0$, $\sigma_Y^2 = 1$, $\sigma_X^2 = 1$, and $v = 30$ (one may continue for others choices of parameters). Afterwards, for a given value of design parameters $G, D, k-m, k, n$ and ρ_{YX} at $\alpha =$

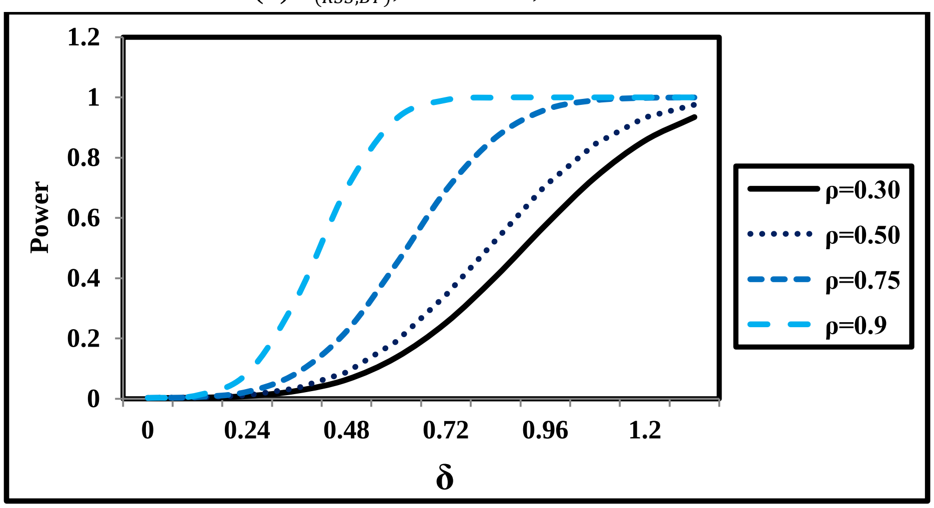
0.0027, pick the required control limits factors given in Tables 4.1-4.3, and construct the control limits (given in Equation 4.2). In the next step, for a given value of design parameters and δ, generates the random samples from bivariate distribution D (such as BN, BT and BLN) and check whether the value of test statistic is inside or outside the control limits ($LCL_{V_{(G,D)}}$, $UCL_{V_{(G,D)}}$). The following procedure is repeated 10^5 for varying values of the design parameters with different amount of δ and finally, proportions of the test statistic beyond the control limits ($LCL_{V_{(G,D)}}$, $UCL_{V_{(G,D)}}$) are displayed in the form of Figure 4.1-4.4.

Figure 4.1. Power curves of $\boldsymbol{V_{(G,D)}}$ control charts for a given value of $\boldsymbol{G, D, k-m, k}$ **and** $\boldsymbol{n}$ with an increase in $\boldsymbol{\rho_{YX}}$ at $\boldsymbol{\alpha = 0.0027}$

(a) $V_{(RSS,BN)}$, $k-m=1$, $k=1$ and $n=8$



(b) $V_{(RSS,BT)}$, $k-m=2$, $k=3$ and $n=6$



(c) $V_{(RSS,BT)}$, $k-m=2$, $k=4$ and $n=6$

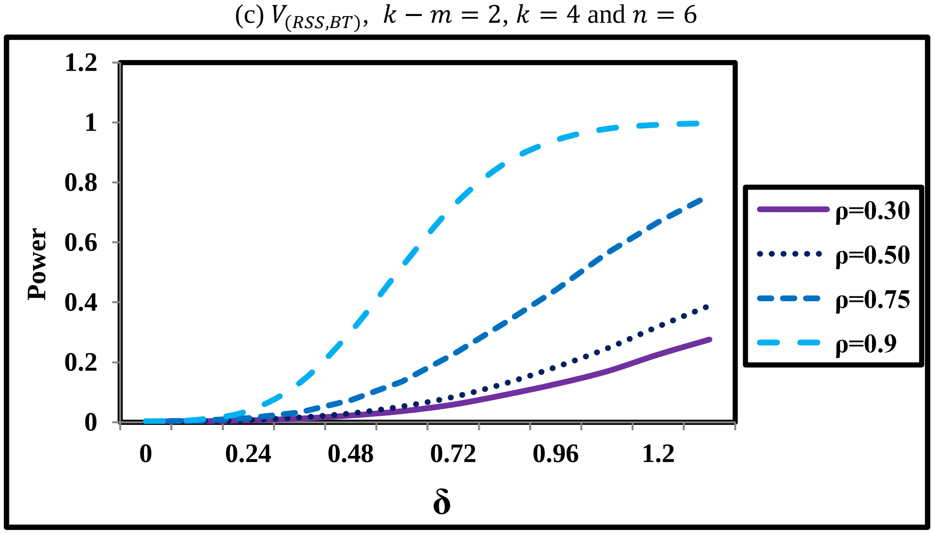
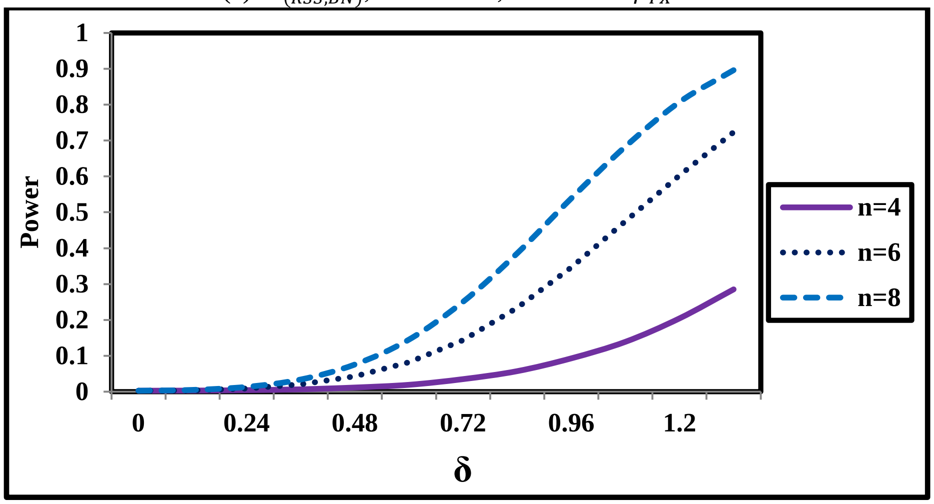
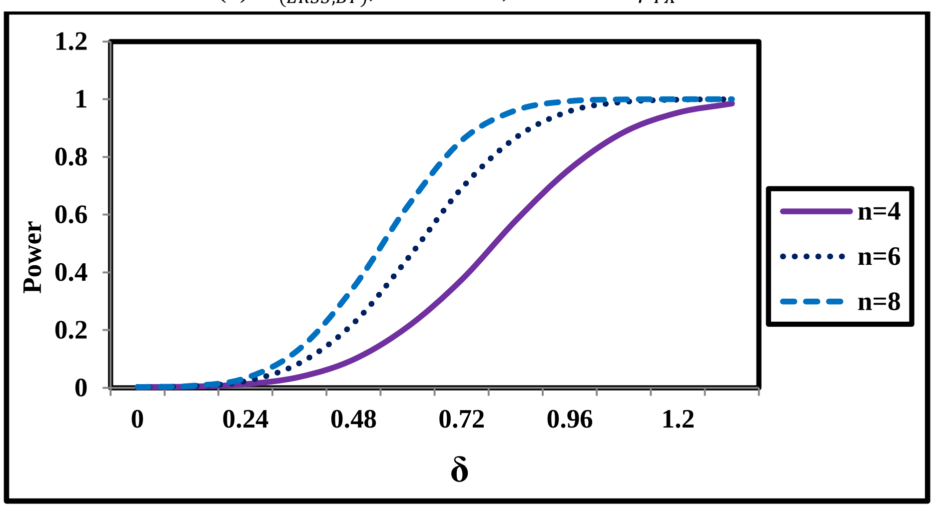


Figure 4.2. Power curves of $\boldsymbol{V_{(G,D)}}$ control charts for a given value of $\boldsymbol{G, D, k-m, k}$ **and** $\boldsymbol{\rho_{YX}}$ with different sample sizes $\boldsymbol{n}$ at $\boldsymbol{\alpha = 0.0027}$.

(a) $V_{(RSS,BN)}$, $k-m=1$, $k=1$ and $\rho_{YX}=0.50$



(b) $V_{(ERSS,BT)}$, $k-m=2$, $k=3$ and $\rho_{YX}=0.75$



(c) $V_{(DERSS,BLN)}$, $k-m=2$, $k=4$ and $\rho_{YX}=0.30$

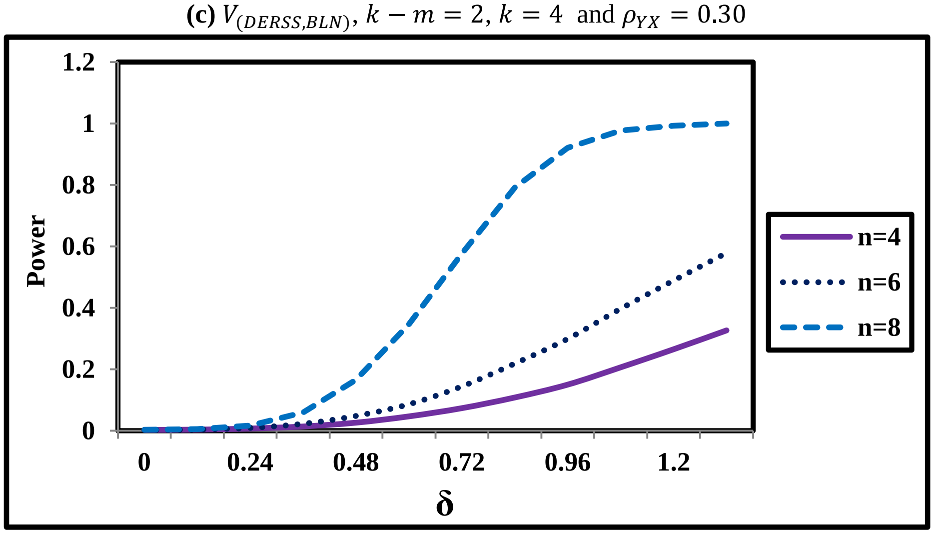


Figure 4. 3. Power curves of of $\boldsymbol{V_{(G,D)}}$control charts for a given value of $\boldsymbol{G, D, n}$ **and** $\boldsymbol{\rho_{YX}}$ with different choices of $\boldsymbol{k-m}$ and $\boldsymbol{k}$at $\boldsymbol{\alpha = 0.0027}$.

(a) $V_{(RSS,BN)}, n = 4$ and $\rho_{YX} = 0.75$

(b) $V_{(ERSS,BT)}, n = 6$ and $\rho_{YX} = 0.50$

(c) $V_{(DERSS,BN)}, n = 6$ and $\rho_{YX} = 0.50$

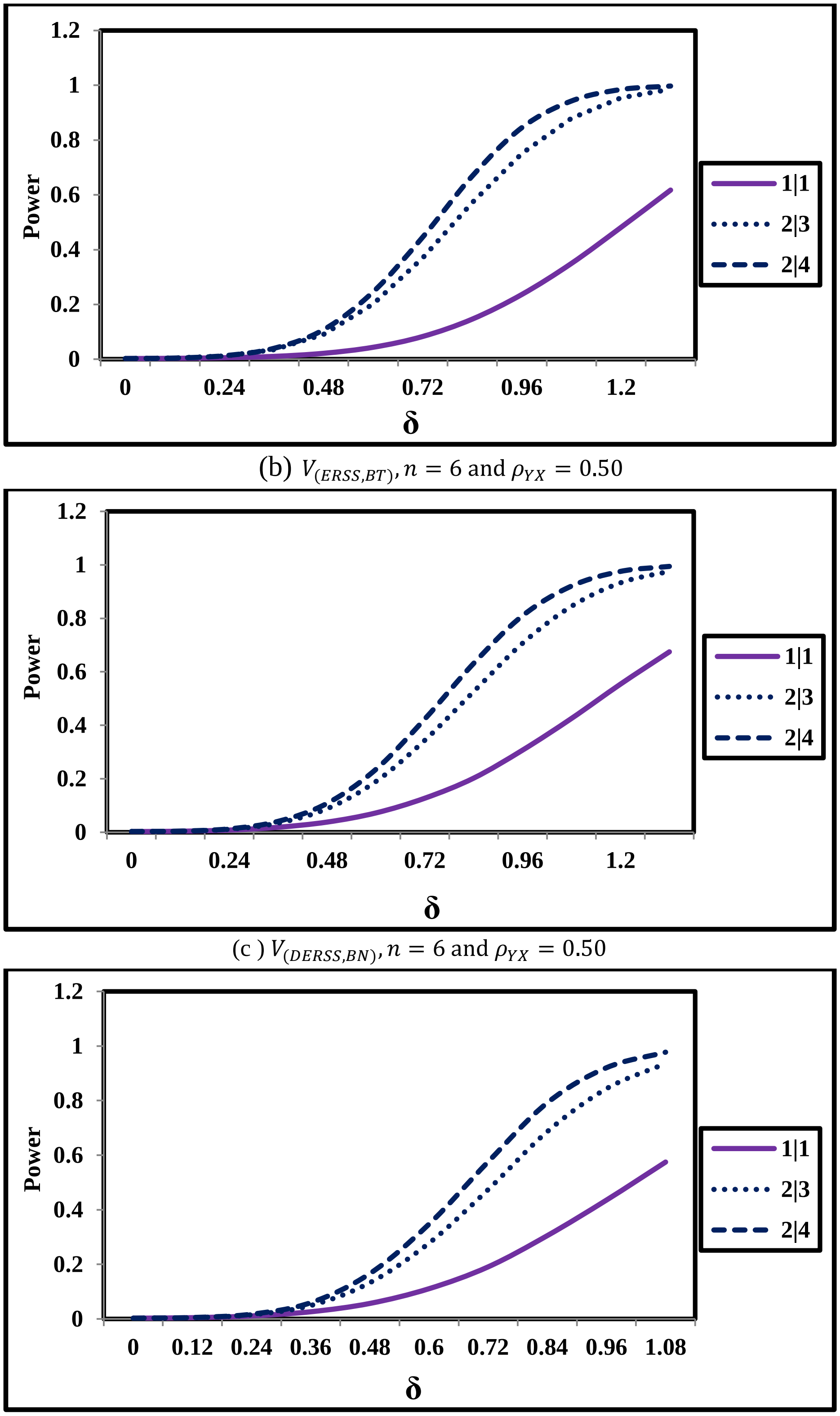
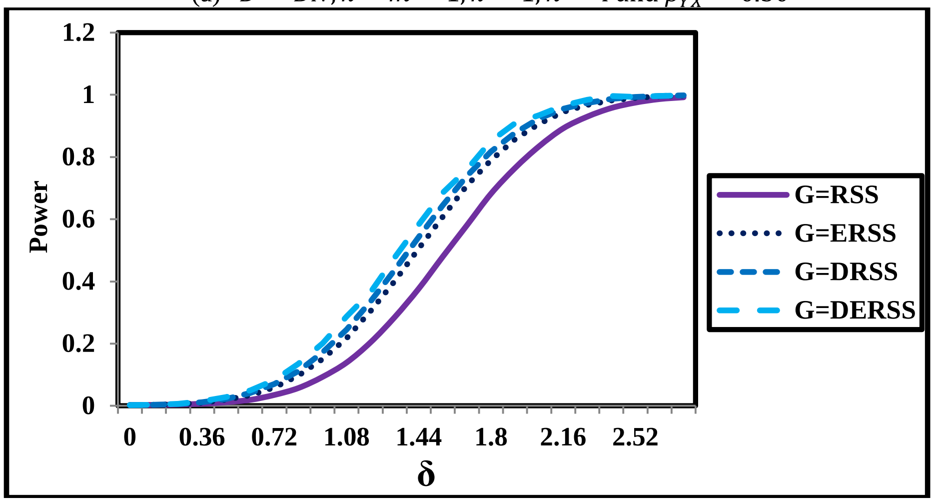
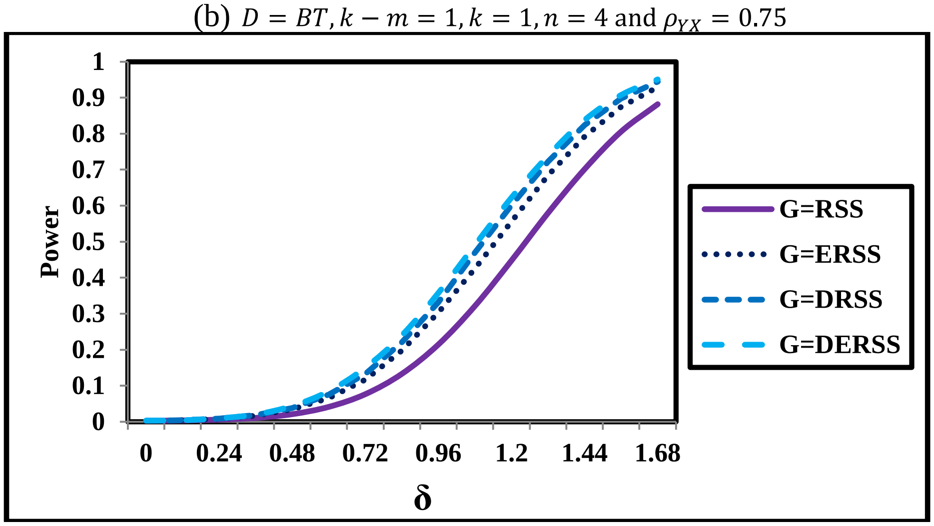


Figure 4. 4. Power curves of of $\boldsymbol{V_{(G,D)}}$ control charts for a given value of $\boldsymbol{D, k-m, k, n}$ **and** $\boldsymbol{\rho_{YX}}$ with different choices $\boldsymbol{G}$ at $\boldsymbol{\alpha = 0.0027}$.

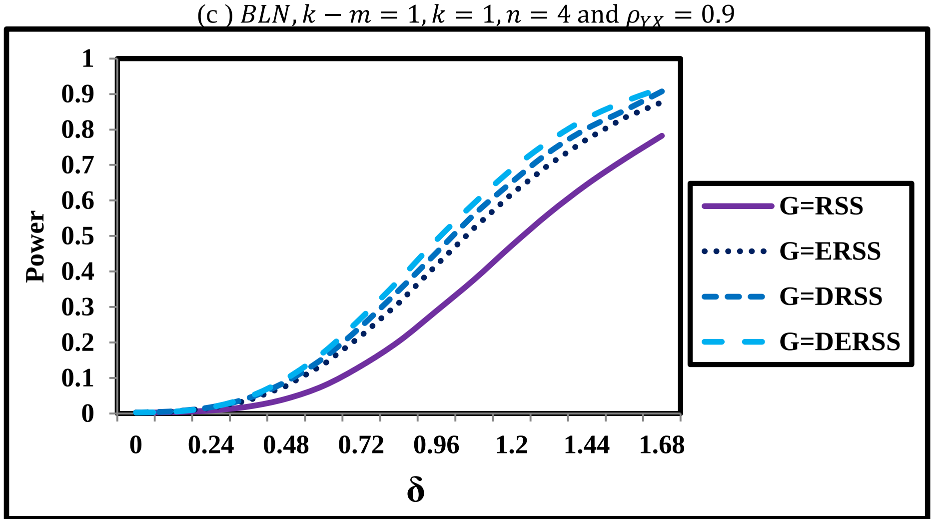
(a) $D = BN, k - m = 1, k = 1, n = 4$ and $\rho_{YX} = 0.50$



(b) $D = BT, k - m = 1, k = 1, n = 4$ and $\rho_{YX} = 0.75$



(c) $BLN, k - m = 1, k = 1, n = 4$ and $\rho_{YX} = 0.9$



4.4.2 Performance evaluation of the $V_{(G,SC)}$ control charts

For evaluating the performance of $V_{(G,SA)}$ control charts in term of false alarm rate, we consider bivariate skewed and heavy tailed distributions which include bivariate gamma and bivariate lognormal, and different contaminated situations. For contaminated environment we consider localized variance disturbance (LVD) and localized mean disturbance (LMD). The details of LVD and LMD can be seen in the article of Tatum (1997), and Schoonhoven et al., (2011). Moreover, in case of LVD, 95% probability of each sample being drawn from a bivariate lognormal distribution (represents a heavy tailed distribution) with scale matrix is $\Sigma = \begin{bmatrix} \sigma^2_{Y(s)} & \sigma_{Y(s)}\sigma_{X(s)}\rho_{YX} \\ \sigma_{Y(s)}\sigma_{X(s)}\rho_{YX} & \sigma^2_{X(s)} \end{bmatrix}$, and 5% probability of being drawn from $\Sigma = \begin{bmatrix} a^2\sigma^2_{Y(s)} & a\sigma_{Y(s)}\sigma_{X(s)}\rho_{YX} \\ a\sigma_{Y(s)}\sigma_{X(s)}\rho_{YX} & \sigma^2_{X(s)} \end{bmatrix}$, where $a > 1$ is the amount of disturbance in the variance of study variable, $\sigma^2_{Y(s)}$ and $\sigma^2_{X(s)}$ are variances of the scale matrix. Likewise, in case of LVD, 95% probability of each sample being drawn from a given bivariate lognormal distribution with the given location vector $\mu = ln\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 5% probability of being drawn from $\mu = ln\begin{bmatrix} b \\ 1 \end{bmatrix}$ respectively, where $b > 1$ is the amount of disturbances in the mean of study variable. Remaining steps for computing the false alarm rate is as follows: Step 1: for a given value of design parameters G, k_3, n and ρ_{YX} at $\alpha = 0.0027$, pick the required control limits factors given in Tables 4.6, and construct the control limits (given in Equation 4.4). Step 2: 10^5 random samples of size n are generated from skewed distribution, heavy tailed distribution, and their contaminated environments for a given value of $G, k_3, n\,, \rho_{YX}$, a

and b. Step 3: proportion of test statistics ($V_{G,j}, j = 1,2,3, \ldots r$) going outside the control limits ($LCL_{V_{(G,D)}}$,$UCL_{V_{(G,D)}}$) is considered as false alarm rate.

For comparison purpose, we have also computed the false alarm rate of existing dual auxiliary information based on control charts (see, section 4.3.1), and some proposed control charts (such as $V_{(DRSS,BN)}$, and $V_{(DERSS,BN)}$) control chart by considering the skewed distribution, heavy tailed distribution and their contamination cases. We have provided results in Tables 4.7-4.9 for some selective choices of k_3, n , ρ_{YX}, a, and b at $\alpha = 0.0027$.

Table 4.7 .False alarm rate of $\boldsymbol{V}_{(G,SC)}$ and $\boldsymbol{V}_{(G,BN)}$control charts under skewed and heavy tailed distribution with varying values of $\boldsymbol{\rho_{YX}}$ $\boldsymbol{n}, \boldsymbol{k_3}$ and $\boldsymbol{G}$ at $\boldsymbol{\alpha = 0.0027}$.

$k_3 = 0.8$										
ρ_{YX}	Distributions	Methods	RSS		ERSS		DRSS		DERSS	
			n=5	n=7	n=5	n=7	n=5	n=7	n=5	n=7
0.50	Lognormal	$V_{(G,SC)}$	0.0041	0.0026	0.0032	0.0031	0.0038	0.0042	0.0033	0.0040
		$V_{(G,BN)}$	0.0050	0.0028	0.0044	0.0034	0.0048	0.0042	0.0036	0.0026
	Gamma	$V_{(G,SC)}$	0.0034	0.0028	0.0035	0.0028	0.0030	0.0038	0.0022	0.0022
		$V_{(G,BN)}$	0.0042	0.0032	0.0045	0.0038	0.0030	0.0038	0.0032	0.0032
0.75	Lognormal	$V_{(G,SC)}$	0.0034	0.0026	0.0028	0.0028	0.0027	0.0044	0.0041	0.0050
		$V_{(G,BN)}$	0.0040	0.0040	0.0034	0.0026	0.0036	0.0036	0.0026	0.0026
	Gamma	$V_{(G,SC)}$	0.0041	0.0031	0.0025	0.0028	0.0035	0.0028	0.0028	0.0018
		$V_{(G,BN)}$	0.0042	0.0031	0.0030	0.0034	0.0035	0.0031	0.0028	0.0019
0.90	Lognormal	$V_{(G,SC)}$	0.0042	0.0027	0.0032	0.0018	0.0030	0.0022	0.0032	0.0050
		$V_{(G,BN)}$	0.0046	0.0036	0.0035	0.0026	0.0030	0.0031	0.0021	0.0031
	Gamma	$V_{(G,SC)}$	0.0035	0.0022	0.0026	0.0021	0.0036	0.0030	0.0025	0.0014
		$V_{(G,BN)}$	0.0036	0.0033	0.0026	0.0021	0.0042	0.0036	0.0024	0.0017

$k_3 = 2$										
ρ_{YX}	Distributions	Methods	RSS		ERSS		DRSS		DERSS	
			n=5	n=7	n=5	n=7	n=5	n=7	n=5	n=7
0.50	Lognormal	$V_{(G,SC)}$	0.0070	0.0054	0.0050	0.0052	0.0046	0.0040	0.0056	0.0100
		$V_{(G,BN)}$	0.0102	0.0082	0.0089	0.0062	0.0098	0.0067	0.0062	0.0040
	Gamma	$V_{(G,SC)}$	0.0062	0.0038	0.0037	0.0024	0.0032	0.0036	0.0017	0.0019
		$V_{(G,BN)}$	0.0094	0.0068	0.0084	0.0055	0.0080	0.0050	0.0064	0.0048
0.75	Lognormal	$V_{(G,SC)}$	0.0056	0.0047	0.0052	0.0038	0.0054	0.0040	0.0045	0.0078
		$V_{(G,BN)}$	0.0086	0.0076	0.0072	0.0049	0.0078	0.0067	0.0054	0.0040
	Gamma	$V_{(G,SC)}$	0.0062	0.0051	0.0050	0.0014	0.0041	0.0036	0.0016	0.0010
		$V_{(G,BN)}$	0.0072	0.0064	0.0063	0.0033	0.0058	0.0050	0.0034	0.0026
0.90	Lognormal	$V_{(G,SC)}$	0.0132	0.0052	0.0042	0.0037	0.0040	0.0026	0.0036	0.0040
		$V_{(G,BN)}$	0.0065	0.0061	0.0059	0.0048	0.0064	0.0050	0.0039	0.0026
	Gamma	$V_{(G,SC)}$	0.0163	0.0050	0.0039	0.0016	0.0044	0.0025	0.0010	0.0004
		$V_{(G,BN)}$	0.0062	0.0050	0.0040	0.0018	0.0045	0.0039	0.0018	0.0010

$k_3 = 2.4$										
ρ_{YX}	Distributions	Methods	RSS		ERSS		DRSS		DERSS	
			n=5	n=7	n=5	n=7	n=5	n=7	n=5	n=7
0.50	Lognormal	$V_{(G,SC)}$	0.0096	0.0057	0.0079	0.0056	0.0070	0.0046	0.0094	0.0168
		$V_{(G,BN)}$	0.0124	0.011	0.011	0.0061	0.0115	0.0104	0.0084	0.0065
	Gamma	$V_{(G,SC)}$	0.0104	0.0046	0.0056	0.0025	0.0052	0.0032	0.0021	0.0024
		$V_{(G,BN)}$	0.0121	0.0099	0.0106	0.0060	0.0120	0.0086	0.0078	0.0064
0.75	Lognormal	$V_{(G,SC)}$	0.0071	0.0044	0.0056	0.0048	0.0054	0.0041	0.0050	0.0082
		$V_{(G,BN)}$	0.0102	0.0094	0.0080	0.0050	0.0099	0.0081	0.0063	0.0036
	Gamma	$V_{(G,SC)}$	0.0134	0.0058	0.0054	0.0014	0.0066	0.0034	0.0015	0.0006
		$V_{(G,BN)}$	0.0120	0.0064	0.0067	0.0030	0.0070	0.0060	0.0046	0.0024
0.90	Lognormal	$V_{(G,SC)}$	0.0045	0.0044	0.0053	0.0040	0.0038	0.0039	0.0034	0.0038
		$V_{(G,BN)}$	0.0075	0.0052	0.0068	0.0044	0.0060	0.0057	0.0050	0.0036
	Gamma	$V_{(G,SC)}$	0.0107	0.0050	0.0046	0.0013	0.0065	0.0032	0.0010	0.0003
		$V_{(G,BN)}$	0.0088	0.0048	0.0050	0.0026	0.0066	0.0044	0.0018	0.0007

Table 4.8. False alarm rate of $\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$ and $\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$ control charts under contaminated (localized variance disturbances) heavy tailed distribution (bivariate lognormal) with varying values of $\boldsymbol{\rho}_{\boldsymbol{YX}}$ $\boldsymbol{n}, \boldsymbol{k}_{\boldsymbol{3}}$ and $\boldsymbol{G}$ at $\boldsymbol{\alpha = 0.0027}$.

$k_3 = 2$ and $a = 1.50$						
$\boldsymbol{\rho}_{\boldsymbol{YX}}$	$\boldsymbol{n}$	Methods	RSS	ERSS	DRSS	DERSS
0.5	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0080	0.0100	0.0121	0.0097
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0170	0.0150	0.0167	0.0130
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0088	0.0107	0.0089	0.0153
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0153	0.0125	0.0152	0.0100
0.75	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0113	0.0104	0.0084	0.0114
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0157	0.0153	0.0136	0.0162
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0100	0.0114	0.0093	0.0156
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0141	0.0143	0.0150	0.0157
$k_3 = 2.4$ and $a = 1.20$						
0.5	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0113	0.0112	0.0070	0.0067
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0119	0.0115	0.0136	0.0093
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0080	0.0062	0.0060	0.0158
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0130	0.0094	0.0112	0.0059
0.75	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0076	0.0060	0.0054	0.0050
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0113	0.0097	0.0098	0.0072
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0068	0.0057	0.0042	0.0100
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0098	0.0079	0.0085	0.0070

Table 4.9. False alarm rate of $\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$ and $\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$ control charts under contaminated (localized mean disturbances) heavy tailed distribution (bivariate lognormal) with varying values of $\boldsymbol{\rho}_{\boldsymbol{YX}}$ $\boldsymbol{n}, \boldsymbol{k}_{\boldsymbol{3}}$ and $\boldsymbol{G}$ at $\boldsymbol{\alpha = 0.0027}$.

$k_3 = 2$ and $b = 1.50$						
$\boldsymbol{\rho}_{\boldsymbol{YX}}$	$\boldsymbol{n}$	Methods	RSS	ERSS	DRSS	DERSS
0.5	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0071	0.0080	0.0054	0.0077
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0122	0.0118	0.0120	0.0096
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0048	0.0053	0.0051	0.0110
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0099	0.0084	0.0116	0.0062
0.75	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0059	0.0056	0.0050	0.0055
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0112	0.0099	0.0110	0.0081
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0048	0.0058	0.0052	0.0072
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0107	0.0083	0.0096	0.0063
$k_3 = 2.4$ and $b = 1.20$						
0.5	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0090	0.0077	0.0069	0.0062
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0118	0.0106	0.0131	0.0088
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0074	0.0060	0.0055	0.0157
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0132	0.0092	0.0133	0.0063
0.75	5	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0085	0.0063	0.0060	0.0053
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0137	0.0120	0.0110	0.0085
	7	$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{SC})}$	0.0046	0.0059	0.0050	0.0089
		$\boldsymbol{V}_{(\boldsymbol{G},\boldsymbol{BN})}$	0.0105	0.0098	0.0116	0.0066

4.5 Results and Discussion

We have observed the following results based on the Figures 4.1-4.4 and Tables 4.7-4.9:

i. Performance of $V_{(G,D)}$ control charts is increasing with an increase in $k - m, k, n, \rho_{YX}$ and δ (see, Figures 4.1-4.3 and Tables 4.4-4.6). The following properties of proposed control charts are in accordance with several studies such as Riaz (2008a), Mehmood et al. (2013b), and Mehmood et al. (2014).

ii. Attaching more runs rules with the design structures of $V_{(G,D)}$ control charts are helpful for detection of smaller and moderate shifts (see, Figure 4.3). Also, behaviour of different runs rules is in accordance with Riaz et al. (2011).

iii. Among different sampling strategies double ranked set strategies based $V_{(G,D)}$control charts have best performance compared to single ranked set strategies based control charts in general (see, Figure 4.4). Similar findings were given by Abujiya & Muttlak (2004) , Mehmood et al. (2013b), and Mehmood et al. (2014) in their articles. Moreover, among single ranked set strategies ERSS based control chart are founded at 1st position followed by RSS (cf. Abbasi & Riaz, 2015) whereas in double ranked set strategies, DERSS declared at 1st position followed by DRSS.

iv. Performance of $V_{(G,D)}$control charts with varying sampling strategies and runs rules comes closer to each other for large sample size n, but in most of the practices we have limited sample size (such as $n = 4$, and $n = 6$).

v. The efficient design structures given in Equation (4.2) can be considered as the generalized form of other existing studies such as Abbasi & Riaz (2015), when $G = (RSS \text{ and } ERSS), D = BN, k - m = 1, k = 1$ (see, Figures 4.1-4.3). Moreover, we have improved the design structures of existing dual auxiliary information based on control charts for the detection of smaller shifts (by incorporating the runs rules), but also extended their design structure through other process distributions and double ranked set strategies.

vi. When design structure of $V_{(G,D)}$ control charts based on a known symmetrical distribution (such as bivariate normal), but an ongoing process follows a skewed distribution or contaminated skewed distribution, then in such situation extreme ranked set strategies based control charts are comparatively more robust than ranked set strategies based control charts. It is important to mention here, for skewed distribution (such as bivariate gamma) with large sample size and high skewness level, false alarm rate of double extreme ranked set strategies based control charts is noticeably smaller than other ranked set based on control charts (see, Table 4.7-4.9).

vii. Skewness correction control charts $V_{(G,SC)}$ are more robust than known symmetrical distribution based on control charts, when process distributions are skewed and heavy tailed (with small sample size $n = 5$). Among different skewness correction control charts, extreme ranked set strategies based control charts are ranked at 1st position due to their ability towards maintaining the false alarm rate followed by ranked set strategies based control charts especially, for high skewness ($k_3 \geq 2$), small sample size ($n = 5$) and $\rho_{YX} > 0.5$. Furthermore, for large sample size ($n \geq 7$), ranked set strategies based $V_{(G,SC)}$ control charts ranked at 1st position followed by extreme ranked set strategies based control charts (see, Tables 4.7-4.9).

viii. When sample size is large ($n > 7$), $V_{(DERSS,BN)}$ control chart is relatively more robust than $V_{(G,SC)}$ control charts (see, Table 4.7). But, it is a fact that in most of the practices practitioner prefer small sample with high precision.

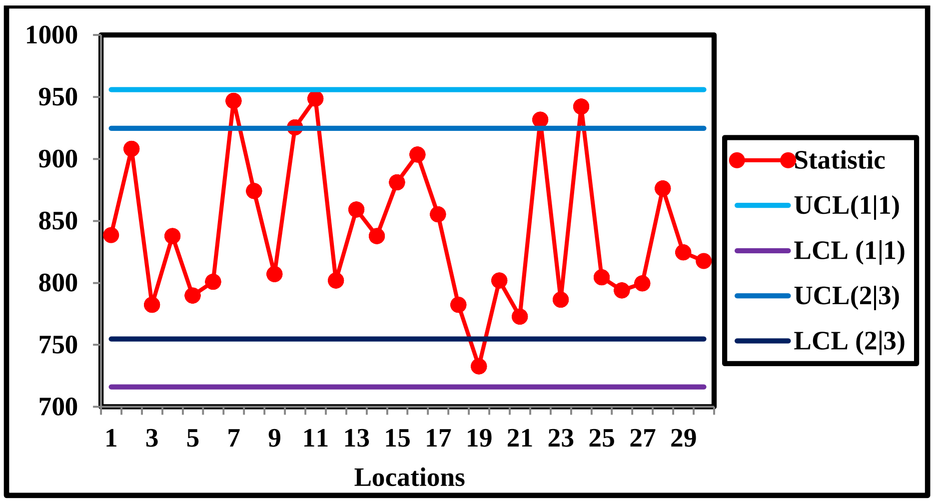
4.6 Application

In this section we provide a real life application of proposed control charts for monitoring the stability of physico-chemical parameters of groundwater. The stability of ground water parameters is always considered important for an industrial process, crop yield and for drinking water which all ultimately affects the industrial production, production of a crop yield and human health respectively. More specifically crop yield depends on numbers of factors such as color, acidity, hardness, PH , temperature, and sulphite. In order to show the application of proposed control charts we consider two physico-chemical parameters of groundwater, which include total dissolved solids and total hardness of water. In more detail, Total dissolved solids is considered as a study variable Y and measured in term of electric conductivity (EC)), whereas total hardness of water is considered as auxiliary variable X and measured in term of calcium magnesium carbonates.

In order to show the significance of proposed location control charts, we consider groundwater (used for irrigation of crop yield) of District Rahim Yar Khan, Pakistan. In more details, we randomly selected 30 locations and then from each location a sample of size 5 is gathered under extreme ranked set sampling (procedure of ERSS can be seen in Mehmood et al. (2013b) and Abbasi & Riaz. (2015). The actual measurements of electric conductivity and calcium magnesium carbonates are tabulated in the form of Table 4.10. Before implementing the control charts we calculated the statistic ($V_{ERSS,j}, j = 1,2,3, \dots r$) using the data set given in Table 4.10 with known in-control parameters are $\mu_Y = 836.06, \mu_X = 4.93, \sigma_Y^2 = 1000, \sigma_X^2 = 1.53$, and $\rho_{YX} = 0.50$.

After calculating the charting statistic we have implemented $V_{(ERSS,BN)}$ control chart with two runs rules (1|1 and 2|3) for monitoring the variation of each water sample of a given location with respect to electric conductivity (see, Figure 4.5).

Figure 4.5. Monitoring the location parameter of electric conductivity through control charts



From Figure 4.5 it is clear that no signal is detected by first point decision rule ($k - m|k = 1|1$), whereas two signals are triggered by $k - m|k = 2|3$. This shows that attaching more runs rules with the design structure of usual auxiliary information based on control charts boost the performance of the control chart. The following outcomes are in accordance with results and discussion (see, section 4.5).

Table 4. 10. Actual measurement of electric conductivity Y and calcium-magnesium carbonates X

Location (j)		Observation (i)				
		1	2	3	4	5
1	$Y_{[i,erss],1}$	860	817	880	912	856
	$X_{(i,erss),1}$	6.4	5.8	4.4	6.5	4.8
2	$Y_{[i,erss],2}$	846	830	845	890	897
	$X_{(i,erss),2}$	3.8	3.5	3.6	3.8	4.2
3	$Y_{[i,erss],3}$	850	828	879	803	887
	$X_{(i,erss),3}$	7.5	6.3	6	6.2	6.9
4	$Y_{[i,erss],4}$	806	835	780	790	757
	$X_{(i,erss),4}$	4.7	2.7	3	3.7	5.1
5	$Y_{[i,erss],5}$	750	792	760	720	791
	$X_{(i,erss),5}$	4.2	5.8	4.6	3.7	3
6	$Y_{[i,erss],6}$	744	720	790	775	782
	$X_{(i,erss),6}$	3.2	3.6	3.5	4.9	4.7
7	$Y_{[i,erss],7}$	870	810	840	815	820
	$X_{(i,erss),7}$	1.9	1.8	1.7	2	3
8	$Y_{[i,erss],8}$	888	825	860	880	895
	$X_{(i,erss),8}$	4.5	5.1	4.8	4.7	5
9	$Y_{[i,erss],9}$	775	792	750	812	742
	$X_{(i,erss),9}$	5	4	4.9	3	3.7
10	$Y_{[i,erss],10}$	868	885	885	900	860
	$X_{(i,erss),10}$	3.5	3.8	4	4.5	3.2
11	$Y_{[i,erss],11}$	825	810	825	850	830
	$X_{(i,erss),11}$	2.5	1.8	1.7	1.8	2
12	$Y_{[i,erss],12}$	792	811	816	845	870
	$X_{(i,erss),12}$	5.8	5.9	4.7	5.3	6
13	$Y_{[i,erss],13}$	933	870	909	933	925
	$X_{(i,erss),13}$	6	6.5	6	6.3	6.6
14	$Y_{[i,erss],14}$	910	909	888	933	860
	$X_{(i,erss),14}$	5.3	7.2	7.3	6.3	6.2
15	$Y_{[i,erss],15}$	950	960	927	990	890
	$X_{(i,erss),15}$	7	6.3	6.6	7.3	5.1
16	$Y_{[i,erss],16}$	967	960	914	895	935
	$X_{(i,erss),16}$	5	6	4.5	6.5	6.4
17	$Y_{[i,erss],17}$	920	909	867	890	945
	$X_{(i,erss),17}$	6.4	6.2	5.2	5.9	7.2
18	$Y_{[i,erss],18}$	850	828	879	803	887
	$X_{(i,erss),18}$	7.5	6.3	6	6.2	6.9
19	$Y_{[i,erss],19}$	760	725	790	795	750
	$X_{(i,erss),19}$	5.7	5.5	5.9	5.8	5.6
20	$Y_{[i,erss],20}$	781	740	798	803	812
	$X_{(i,erss),20}$	4	3.6	5	5	5.2
21	$Y_{[i,erss],21}$	870	845	835	828	773
	$X_{(i,erss),21}$	6.8	6.3	6.4	6.2	6
22	$Y_{[i,erss],22}$	858	873	880	900	820
	$X_{(i,erss),22}$	3.1	3.3	3	3.5	3.7
23	$Y_{[i,erss],23}$	745	773	730	780	732
	$X_{(i,erss),23}$	3.8	4.8	3.6	3.9	4.3
24	$Y_{[i,erss],24}$	990	933	940	980	933
	$X_{(i,erss),24}$	.	6	4.7	4	6.3
25	$Y_{[i,erss],25}$	872	907	914	830	856
	$X_{(i,erss),25}$	7	6.7	7.3	6.4	6
26	$Y_{[i,erss],26}$	830	780	867	820	825
	$X_{(i,erss),26}$	5.7	5.5	5.9	5.7	5.6
27	$Y_{[i,erss],27}$	750	790	860	860	810
	$X_{(i,erss),27}$	5.8	4.8	5.5	5.3	5
28	$Y_{[i,erss],28}$	880	840	867	909	867
	$X_{(i,erss),28}$	5	4.3	4.9	5.2	4.8
29	$Y_{[i,erss],29}$	880	918	915	890	840
	$X_{(i,erss),29}$	6.3	6.8	6.6	6.8	6
30	$Y_{[i,erss],30}$	747	730	730	720	790
	$X_{(i,erss),30}$	2.9	3.6	2.2	3.6	3.2

4.7 Concluding remarks

In the current chapter we considered regression estimators under different ranked set strategies. These estimators utilized the auxiliary information at both stages instead of using it either for ranking the units or estimation. Based on these estimators we proposed location control charts for known process distributions and unknown skewed process distributions under different sampling strategies. We have investigated the performance of proposed control charts using probability to signals and false alarm rate by taking into accounts skewed and heavy tailed distribution along with their contaminated cases.

The results of the current chapter indicated that control charts based on extreme ranked set strategies have higher signaling probability than control charts based on ranked set strategies. Moreover, design structures of extreme ranked set strategies based on control charts under known symmetrical distribution are more robust than ranked set strategies based on control charts for skewed, heavy tailed and contaminated distributions.

Likewise we have investigated the performance of skewness correction control charts$V_{(G,SC)}$. The results revealed that $V_{(G,SC)}$control charts are more robust for skewed and heavy tailed processes (especially for small sample size) compared to known symmetrical distribution based on control charts. Our further analysis cleared that among different $V_{(G,SC)}$ control charts, extreme ranked set strategies based on control charts (with high skewness) are ranked at 1st position for small sample size followed by ranked set strategies based on control charts for large sample size. Lastly, real life example is also in favor of the subject of the current article.

CHAPTER 5

ON THE EFFECTIVE DUAL USE OF AUXILIARY INFORMATION IN DISPERSION CONTROL CHARTS

During the last decade, variance control charts based on different sampling schemes have attracted research interest in the field of statistical process control (SPC). These charts used extra (auxiliary) information either for ranking of units or estimation rather than using it for both. The effectiveness of a control chart can be increased by utilizing the auxiliary information for dual purposes. This chapter is focused on developing a generalized structure of variance control charts based on dual use of auxiliary information under different sampling strategies and runs rules. The generalized structure mainly depends on three auxiliary information based estimators with dual use of auxiliary information, three bivariate process distributions and variety of sampling schemes. The performance of the proposed control charts is investigated by assessing the power curve. We have observed that the proposals of the study perform better than its complement. An application example is also provided for practitioners concerns to monitor the stability of physico-chemical parameter of groundwater.

5.1 Introduction

Control charts are used to diagnose the special cause of variation in the process parameters. The special cause of variation is also termed as assignable or non-random variations. An extra (auxiliary) information on some additional process variables may add value to the monitoring of process parameters. It may be in the form of ranking the units of interest before getting an actual measurement or estimating the parameter(s) of interest. Some of the efficient uses of auxiliary information at sampling stage may be seen in Muttlak & Al-Sabah (2003), Abujiya & Muttlak (2004), and Mehmood et al. (2013b). They proposed ranked set schemes based control charts for the monitoring of location parameter by considering the auxiliary information to rank the units of interest. Likewise, Mehmood et al. (2014) utilized auxiliary information and developed dispersion control charts under different sampling schemes. They concluded that dispersion control charts based on different ranked set strategies have superior performance compared to the SRS.

Another popular mechanism for utilizing the auxiliary information has been seen at the estimation stage for designing a control chart. Riaz, (2008a), and Riaz & Does (2009) considered the auxiliary information at estimation stage and proposed process variability control chart under SRS for the monitoring of dispersion parameter. Moreover, Riaz et al. (2014) and Ahmad et al. (2014) developed the variance control charts under the normality assumption by utilizing the auxiliary information at the estimation stage in the form of ratio, ratio exponential, power ratio, ratio regression, power ratio regression, and ratio exponential regression estimators. They observed that control charts based on regression-type estimators have attractive detection abilities for an efficient monitoring of process parameters.

Yu & Lam (1997), and Muttlak (2001) recommended the use of auxiliary information for dual purposes instead of using it either for ranking of units or estimation. They suggested auxiliary information based regression-type estimator under ranked set sampling. Riaz et al. (2014) provided the future recommendation to develop the auxiliary information based on variance control charts under different sampling strategies and runs rules. The same is the subject of our current study.

In this chapter, we mainly intend to design the control charts that use auxiliary information at two stages (ranking and estimation). Our study covers three variance estimators define under three bivariate distribution and variety of sampling strategies. The statistical distributions include bivariate normal (BN) distribution, bivariate t (BT) distribution and bivariate lognormal (BLN) distribution. Furthermore, sampling strategies include simple random sampling, ranked set sampling, and extreme ranked set sampling.

These choices are based on the filtration of Mehmood et al. (2013b), Mehmood et al. (2014), and Riaz et al. (2014).

The organization of rest of the chapter is as follows: In Section 5.2, we discuss the uses of auxiliary information for estimation of variance, which include single and dual use of auxiliary information in the form of different variance estimators, under different sampling strategies and bivariate distributions. In Section 5.3, we propose a generalized structure of variance control charts based on the dual use of auxiliary information, supported by some runs rules. In Section 5.4, we investigate the performance of proposed charts in term of power curve. Section 5.5 elaborate results and discussions based on the performance measure. Section 5.6 presents an application of the proposed charts in groundwater monitoring. Section 5.7 covers the summary and conclusions of the whole study.

5.2 Uses of auxiliary information for estimation of variance

In the following section we provide different estimators of variance by considering different sampling strategies, in which role of auxiliary information can be seen at single stage (either at estimation stage or ranking of units). Then, we merge the role of auxiliary information at both stages in the form of different auxiliary information based on variance estimators, as provided by Yu & Lam (1997), and Muttlak (2001) for location estimators.

5.2.1 Single use of auxiliary information for estimation of variance

The procedure of SRS is illustrated in the following steps as follow:

Step 1: Select a random sample of size n from the population of interest, and measure Y(say study variable) and X(say auxiliary variable).

Step 2: Repeat the above procedure r times, and finally, nr data values of actual measurement are assembled.

Let $\left(\left(Y_{[i,srs],j}, X_{(i,srs),j}\right), i = 1,2,3, \dots, n \text{ and } j = 1,2,3, \dots, r\right)$ denotes jth simple random sample of size , and it may follows any of the bivariate distribution D such as bivariate normal (BN), bivariate t (BT), and bivariate lognormal (BLN), having means μ_Y and μ_X and variances, σ_Y^2 and σ_X^2 respectively. Before providing the variance estimators, it is important to mention here for bivariate lognormal distribution (which is a case of skewed distribution) we consider $Y_{[i,srs],j} = lnQ_{[i,srs]}, X_{[i,srs],j} = lnW_{(i,srs),j}$ with population means and variances are $\mu_Y = l\,n\left(\mu_Q\right) - \left(\frac{\sigma_Q^2}{2}\right),\ \mu_X = l\,n(\mu_W) - \left(\frac{\sigma_W^2}{2}\right), \sigma_Y^2 = ln\left(1 + \frac{\sigma_Q^2}{\mu_Q^2}\right)$

and $\sigma_X^2 = ln\left(1+\frac{\sigma_W^2}{\mu_W^2}\right)$, where $(Q_{[i,srs],j}, W_{(i,srs),j})$ denotes jth simple random sample of size n, and follows bivariate lognormal distribution with means are μ_Q and μ_W , and variances are σ_Q^2 and σ_W^2 respectively. Details of the following transformation are provided by Stedinger et al. (1993), and Yerel & Konuk (2009). Moreover, it is helpful to bring the non-linear random variable such as $(Q_{[i,srs]}, W_{(i,srs),j})$, into linear form $\left(Y_{[i,srs],j}, X_{(i,srs),j}\right)$. Assumption of linearity of a variable is usually considered important for using a difference estimator and regression estimator. Yu & Lam (1997), and Muttlak (2001) concluded that regression estimators works efficiently when $\left(Y_{[i,srs],j}, X_{(i,srs),j}\right)$ is linear. They also concluded some kind of transformation can be used for using the regression estimator when variable of interests are non linear.

Then variance estimator under simple random sampling (see Riaz et al., 2014) for the jth simple random sample of size n and covering the three bivariate distributions ($D = BN$, BT and BLN) are as follow:

$$V^{(1)}_{(SRS,D),j} = s^2_{Y_{(SRS,D),j}} + b_{s^2_{Y_{(SRS,D),j}}\, s^2_{X_{(SRS,D),j}}}\left[\sigma^2_{X(D)} - s^2_{X_{(SRS,D),j}}\right], \qquad j = 1,2,3, \dots r,$$

$$V^{(2)}_{(SRS,D),j} = s^2_{Y_{(SRS,D),j}}\left(\frac{\sigma^2_{X(D)}}{s^2_{X_{(SRS,D),j}}}\right)^{\rho^2_{YX}} + b_{s^2_{Y_{(SRS,D),j}}\, s^2_{X_{(SRS,D),j}}}\left[\sigma^2_{X(D)} - s^2_{X_{(SRS,D),j}}\right],$$

$$V^{(3)}_{(SRS,D),j} = s^2_{Y_{(SRS,D),j}} exp\left(\frac{\sigma^2_{X(D)} - s^2_{X_{(SRS,D),j}}}{\sigma^2_{X(D)} + s^2_{X_{(SRS,D),j}}}\right)$$
$$+ b_{s^2_{Y_{(SRS,D),j}}\, s^2_{X_{(SRS,D),j}}}\left[\sigma^2_{X(D)} - s^2_{X_{(SRS,D),j}}\right],$$

Moreover, mean and variance of $V^{(1)}_{(SRS,D),j}$, $V^{(2)}_{(SRS,D),j}$ and $V^{(3)}_{(SRS,D),j}$ are given below:

$$\mu_{V^{(1)}_{(SRS,D),j}} = \sigma^2_{Y(D)} ,$$

$$\sigma^2_{V^{(1)}_{(SRS,D),j}} = \frac{\sigma^2_{Y(D)}}{n-1}\left[\left(\mu_{40(D)} - 1\right) + \frac{\left(\mu_{04(D)}-1\right)b^2_{(D)}}{R^2_{(D)}} - \frac{2\left(\mu_{22(D)}-1\right)b_{(D)}}{R_{(D)}}\right],$$

$$\mu_{V^{(2)}_{(SRS,D),j}} = \frac{\sigma^2_{Y(D)}}{n-1}\left[\rho^2_{YX}\left\{\left(\mu_{04(D)} - 1\right) - \left(\mu_{22(D)} - 1\right)\right\} + \sigma^2_{Y(D)}\right],$$

$$\sigma^2_{V^{(2)}_{(SRS,D),j}} = \frac{\sigma^4_{Y(D)}}{n-1}\left[\left\{\left(\mu_{40(D)} - 1\right) + \rho^4_{YX}\left(\mu_{04(D)} - 1\right) - 2\rho^2_{YX}\left(\mu_{22(D)} - 1\right)\right\} + \right.$$

$$\left.\frac{b^2_{(D)}\left(\mu_{04(D)}-1\right)}{R^2} - \frac{2b_{(D)}}{R}\left\{\left(\mu_{22(D)} - 1\right) - \rho^2_{YX}\left(\mu_{04(D)} - 1\right)\right\}\right],$$

$$\mu_{V^{(3)}_{(SRS,D),j}} = \frac{\sigma^2_{Y(D)}}{2(n-1)}\left[\left\{\left(\mu_{04(D)} - 1\right) - \left(\mu_{22(D)} - 1\right)\right\} + \sigma^2_{Y(D)}\right],$$

$$\sigma^2_{V^{(3)}_{(SRS,D),j}} = \frac{\sigma^2_{Y(D)}}{n-1}\left[\left(\mu_{40(D)} - 1\right) + 0.25\left(\mu_{04(D)} - 1\right) - 2\left(\mu_{22(D)} - 1\right) + \frac{b^2_{(D)}\left(\mu_{04(D)}-1\right)}{R^2} - \right.$$

$$\left.\frac{2b_{(D)}}{R}\left\{\left(\mu_{22(D)} - 1\right) - 0.5\left(\mu_{04(D)} - 1\right)\right\}\right],$$

where D denotes a bivariate distribution and it can takes any of the value BN , BT and BLN, $s^2_{Y(SRS,D),j}$and $s^2_{X(SRS,D),j}$ are sample variances for the jth SRS based bivariate sample of size n taken from any of the bivariate distribution, ρ_{YX} denotes population correlation coefficient between Y and X, $\mu_{40(D)}$ and $\mu_{04(D)}$are the kurtosis of Y and X respectively for a given value of D, $R_{(D)} = \frac{\sigma^2_{Y(D)}}{\sigma^2_{X(D)}}$, $\mu_{22(D)} = \frac{\eta_{22(D)}}{\sqrt{(\eta_{20(D)})^2(\eta_{02(D)})^2}}$, $\eta_{22(D)} = E(Y-\mu_{Y(D)})^2(X-\mu_{X(D)})^2$,$\eta_{20(D)} = E(Y-\mu_{Y(D)})^2$ and $\eta_{02(D)} = E(X-\mu_{X(D)})^2$. We have provided$\mu_{40(D)}$, $\mu_{04(D)}$, $\mu_{22(D)}$, $\eta_{22(D)}$, $\eta_{20(D)}$, and $\eta_{02(D)}$ in Appendix B by considering the bivariate distribution under consideration. Moreover, $b_{s^2_{Y_{(SRS,D),j}}\,s^2_{X_{(SRS,D),j}}}$ denotes population regression coefficient between $s^2_{Y_{(SRS,D),j}}$ and $s^2_{X_{(SRS,D),j}}$ which depends on the D. We have derived $b_{s^2_{Y_{(SRS,D),j}}\,s^2_{X_{(SRS,D),j}}}$ for bivariate normal and bivariate t distribution (see, Appendix C). It is important to mention here population regression coefficient of bivariate normal and bivariate lognormal (after performing transformation of variables) are approximately equal. The value of $b_{s^2_{Y_{(SRS,D),j}}\,s^2_{X_{(SRS,D),j}}}$ for bivariate normal process, bivariate t and bivariate lognormal are $\rho^2_{YX}\frac{\sigma^2_{Y(BN)}}{\sigma^2_{X(BN)}}$, $\frac{\sigma^2_{Y(BT)}}{\sigma^2_{X(BT)}}\left[\rho^2_{YX}\left(1-\frac{1}{v-1}\right)+\frac{1}{v-1}\right]$, and $\rho^2_{YX}\frac{\sigma^2_{Y(BLN)}}{\sigma^2_{X(BLN)}}$, and respectively. Theoretical justification of providing $b_{s^2_{Y_{(SRS,D),j}}\,s^2_{X_{(SRS,D),j}}}$ for each of the distribution can be seen in the article of Isaki (1983). Moreover, in most of the practices $b_{s^2_{Y_{(SRS,D),j}}\,s^2_{X_{(SRS,D),j}}}$ is unknown. In case of unknown, one can estimate it using sample information. The estimates of $b_{s^2_{Y_{(SRS,D),j}}\,s^2_{X_{(SRS,D),j}}}$ for a given bivariate distribution D are $\hat{b}_{s^2_{Y_{(SRS,BN),j}},s^2_{X_{(SRS,BN),j}}} = r^2_{(YX,BN),j}\frac{s^2_{Y(SRS,BN)j}}{s^2_{X(SRS,BN),j}}$, $\hat{b}_{s^2_{Y_{(SRS,BT),j}},s^2_{X_{(SRS,BT),j}}} = \frac{s^2_{Y(SRS,BT)j}}{s^2_{X(SRS,BT),j}}\left[r^2_{(YX,BT),j}\left(1-\frac{1}{v-1}\right)+\frac{1}{v-1}\right]$, and $\hat{b}_{s^2_{Y_{(SRS,BLN),j}},s^2_{X_{(SRS,BLN),j}}} = r^2_{(YX,BLN),j}\frac{s^2_{Y(SRS,BLN)j}}{s^2_{X(SRS,BLN),j}}$ respectively, where v denotes degree of freedom, $r^2_{(YX,BN),j}$, $r^2_{(YX,BT),j}$, and $r^2_{(YX,BLN),j}$ represent sample correlation coefficient between Y and X for jth simple random sample of size n. In more detail, rest of the quantities like $s^2_{Y_{(SRS,D),j}}$ and $s^2_{X_{(SRS,D),j}}$ are defined as: $s^2_{Y_{(SRS,D),j}} = \frac{\sum_{i=1}^{n}\left[Y_{[i,srs],j}-\bar{Y}_{(SRS,D),j}\right]^2}{n-1}$, $s^2_{X_{(SRS,D),j}} = \frac{\sum_{i=1}^{n}\left[X_{(i,srs),j}-\bar{X}_{(SRS,D),j}\right]^2}{n-1}$,

$r^2_{(YX,D),j} = \frac{n\sum_{i=1}^{n} Y_{[i,srs],j} X_{(i,srs),j} - \sum_{i=1}^{n} Y_{[i,srs],j} \sum_{i=1}^{n} X_{(isrs),j}}{\sqrt{\left[n\sum_{i=1}^{n} Y^2_{[i,srs],j} - \left(\sum_{i=1}^{n} Y_{[i,srs],j}\right)^2\right]\left[n\sum_{i=1}^{n} X^2_{(i,srs),j} - \left(\sum_{i=1}^{n} X_{(i,srs),j}\right)^2\right]}}$, where, $\bar{Y}_{(SRS,D),j} = \frac{\sum_{i=1}^{n} Y_{[i,srs],j}}{n}$, $\bar{X}_{(SRS,D),j} = \frac{\sum_{i=1}^{n} X_{(i,srs),j}}{n}$, $i = 1,2,3,\dots,n$, $j = 1,2,3,\dots r$, and $D = BN, BT, BLN$. $\bar{Y}_{(SRS,D),j}$ and $\bar{X}_{(SRS,D),j}$ are sample means of the jth SRS-based bivariate sample of size n under a given probability distribution D.

5.2.2 Dual use of auxiliary information for estimation of variance

In section 5.2.1 we have illustrated the single use of auxiliary information for estimation and ranking the units separately. In the following section we merge these ideologies and come up with three variance estimators that can utilize the auxiliary information at both stages. Let $\left(\left(Y_{[i,rss],j}, X_{(i,rss),j}\right), i = 1,2,3,\dots,n \text{ and } j = 1,2,3,\dots,r\right)$ denotes jth ranked set sample of size n (procedure of RSS can be seen in section 5.2.1) and it may follows any of the bivariate distribution D such as bivariate normal (BN), bivariate t (BT), and bivariate lognormal (BLN) with means and variances are $\mu_{Y(D)}$, $\mu_{X(D)}$, $\sigma^2_{Y(D)}$, $\sigma^2_{X(D)}$ respectively. Thus a variance estimators based on the dual use of auxiliary information for the jth ranked set sample of size n is follow:

$$V^{(1)}_{(RSS,D),j} = s^2_{Y(RSS,D),j} + b_{s^2_{Y(RSS,D),j}\, s^2_{X(RSS,D),j}}\left[\sigma^2_{X(D)} - s^2_{X(RSS,D),j}\right], \quad j = 1,2,3,\dots r,$$

$$V^{(2)}_{(RSS,D),j} = s^2_{Y(RSS,D),j}\left(\frac{\sigma^2_{X(D)}}{s^2_{X(RSS,D),j}}\right)^{\rho^2_{YX}} + b_{s^2_{Y(RSS,D),j}\, s^2_{X(RSS,D),j}}\left[\sigma^2_{X(D)} - s^2_{X(RSS,D),j}\right],$$

$$V^{(3)}_{(RSS,D),j} = s^2_{Y(RSS,D),j}\, exp\left(\frac{\sigma^2_{X(D)} - s^2_{X(RSS,D),j}}{\sigma^2_{X(D)} + s^2_{X(RSS,D),j}}\right) + b_{s^2_{Y(RSS,D),j}\, s^2_{X(RSS,D),j}}\left[\sigma^2_{X(D)} - s^2_{X(RSS,D),j}\right],$$

where $s^2_{Y_{(RSS,D),j}}$ and $s^2_{X_{(RSS,D),j}}$ are sample variances for the jth ranked set sample of size n, $b_{s^2_{Y_{(RSS,D),j}} s^2_{X_{(RSS,D),j}}}$ denotes population regression coefficients between $s^2_{Y_{(RSS,D),j}}$ and $s^2_{X_{(RSS,D),j}}$ for a given bivariate distribution D. For more detail, rests of the discussion about $b_{s^2_{Y_{(RSS,D),j}} s^2_{X_{(RSS,D),j}}}$ is similar to simple random sampling.

We also define the above estimators for median ranked set sampling and extreme ranked set sampling. For convenience, the above estimators under SRS, RSS, and ERSS are denoted as, $V^{(1)}_{(G,D),j}$, $V^{(2)}_{(G,D),j}$ and $V^{(3)}_{(G,D),j}$, where G denotes a sample strategy and it can take any value SRS, RSS, and ERSS. It is important to mention here, we have also included $G = SRS$ to make the estimators more generalized. Thus, variance estimators covering three bivariate distributions $(D = BN, BT, BLN)$ and based on three sampling strategies ($G = SRS, RSS,$ and $ERSS$) are given below:

$$V^{(1)}_{(G,D),j} = s^2_{Y_{(G,D),j}} + b_{s^2_{Y_{(G,D),j}} s^2_{X_{(G,D),j}}} \left[\sigma^2_{X(D)} - s^2_{X_{(G,D),j}}\right], \qquad j = 1,2,3, \dots r,$$

$$V^{(2)}_{(G,D),j} = s^2_{Y_{(G,D),j}} \left(\frac{\sigma^2_{X(D)}}{s^2_{X_{(G,D),j}}}\right)^{\rho^2_{YX}} + b_{s^2_{Y_{(G,D),j}} s^2_{X_{(G,D),j}}} \left[\sigma^2_{X(D)} - s^2_{X_{(G,D),j}}\right],$$

$$V^{(3)}_{(G,D),j} = s^2_{Y_{(G,,D),j}} exp\left(\frac{\sigma^2_{X(D)} - s^2_{X_{(G,D),j}}}{\sigma^2_{X(D)} + s^2_{X_{(G,D),j}}}\right) + b_{s^2_{Y_{(G,D),j}} s^2_{X_{(G,D),j}}} \left[\sigma^2_{X(D)} - s^2_{X_{(G,D),j}}\right],$$

where $s^2_{Y_{(G,D),j}}$ and $s^2_{X_{(G,D),j}}$ are sample variances for the jth random sample of size n collected under any of the sampling strategy G. The others quantities are defined earlier

5.3 Dual use of auxiliary information in control charts

In this section we develop a set of variance control charts based on the estimators given in section 5.2 and runs rules. The proposed charts are denoted by $V_{(G,D),j}^{(E)}$, where superscript E denotes the estimator which can take 1, 2 or 3, G denotes the sampling strategy and can take one of the following value, SRS, RSS, and ERSS. We define the control charting structures using two setups, probability limit approach and A-Sigma approach. In the subsequent sections firstly we define decision rules (runs rules) and then design control charting structures of $V_{(G,D),j}^{(E)}$ control charts according to the subjects of the study as well as some recommendations and motivations from different studies (see, section 5.1).

5.3.1 Decision rules of proposed control charts

In designing a Shewhart-type control chart one of the most important factor is to define the decision rules in order to declare process out-of-control. The most commonly used decision rule named as one point decision rule and denoted as 1/1 (see Montgomery, 2009))**:** process can be declared out-of-control if single value of test statistic goes either outside the lower control limit or upper control limit. The following decision rule is sufficient for the detection of larger shifts in the process parameter and therefore, for the detection of smaller as well as moderate shifts, one may attach more runs rules with the design structure of Shewhart-type control charts (cf. Riaz et al. 2011; Mehmood et al., 2013b; Mehmood et al., 2014; Ahmad et al., 2014; and Riaz et al., 2011) indicated some issues in the current used runs rules such as biasedness, inflating the false alarm rate and non-monotonicity. They redefined runs rules and implemented on usual Shewhart-type control charts ($\bar{X}$, R, S and S^2) for increasing their detection ability towards smaller and moderate shifts. The application of these rules is not easy to find in already existing studies related to the auxiliary information based on control charts such as Riaz (2008b), Riaz et al. (2009) and Riaz et al. (2014). On the same ideology of Riaz et al. (2011) runs rules are defined for $V_{(G,D)}^{(E)}$ control charts as follows:"Process can be declared out-of-control, if at least $k-m$ values of the test statistic $(V_{(G,D),j}^{(E)}, j=1,2,3\ldots r)$ out of k consecutive values of $V_{(G,D),j}^{(E)}$ either exceed the lower control limit $LCL_{V_{(G,D)}^{(E)}}$ or upper control limit $UCL_{V_{(G,D)}^{(E)}}$ ".The following term can be represented as $k-m|k$ with the condition that $0 \leq m \leq k-1$.

5.3.2 Control limits based on probability limits approach

In this section we propose control charting structures of $V_{(G,D)}^{(E)}$ control charts using the variance estimators (see, section 5.2) and runs rules (see, section 5.3.1). The propose structures mainly consist of lower control limit ($LCL_{V_{(G,D)}^{(E)}}$) and upper control limit ($UCL_{V_{(G,D)}^{(E)}}$), and have ability to incorporate the auxiliary information for dual purpose. Thus, control charting structures are as follow:

$$LCL_{V_{(G,D)}^{(E)}} = L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}\sigma_{Y(D)}^2$$

$$UCL_{V_{(G,D)}^{(E)}} = L_{(E,G,D,k-m,k,n,\rho_{YX},1-\frac{p}{2})}\sigma_{Y(D)}^2, \qquad (5.1)$$

where $L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$ and $L_{(E,G,D,k-m,k,n,\rho_{YX},1-\frac{p}{2})}$ are control limit factors depend on estimator E ,sampling strategy G, distribution D, decision observations in order to declare a process out-of-control $k-m$, total observations in a given rule k , sample size n, correlation ρ_{YX} between Y and X, and probability of single point p of the test statistic

$V_{(G,D),j}^{(E)}$ either breaches the $LCL_{V_{(G,D)}^{(E)}}$ or $UCL_{V_{(G,D)}^{(E)}}$. The value of p can be obtained by solving the expression, $\alpha = \sum_{k-m \leq k} \frac{k!}{(k-m)!m!} p^{k-m}(1-p)^m$, where $0 \leq m \leq k-1$, for a given value of $k-m$, k and false alarm rate (α). In addition, α is the probability of at least $k-m$ values of test statistic $V_{(G,D),j}^{(E)}$ out of k consecutive values either exceed the $LCL_{V_{(G,D)}^{(E)}}$ or $UCL_{V_{(G,D)}^{(E)}}$ given that the process parameter ($\sigma_{Y(D)}^2$) is in-control.

Furthermore, control limits factors are derived through Monte Carlo simulation. For the said purpose we consider different value of design parameters ($E, G, D\ k-m, k, n, \rho_{YX}$), random samples are generated from a given bivariate distribution D and the charting statistic $V_{(G,D),j}^{(E)}$ are calculated 10^5 times. Finally, the control limits factors are obtained by taking the $\left(1-\frac{p}{2}\right)$th and $\left(\frac{p}{2}\right)$th quantiles of the sampling distribution of $V_{(G,D),j}^{(E)}$ for a given value of $E, G, D, k-m, k, n, \rho_{YX}, \frac{p}{2}$. Moreover, in this study we have tabulated control limits factors of different control charts in the form of Tables 5.1-5.6 under varying values of design parameters ($E, G, D, k-m, k, n, \rho_{YX}$) at $\alpha = 0.0027$. Similarly, control limits factors can be derived for the other choice of design parameters and α.

5.3.3 Control limits based on A-sigma approach

Shewhart (1931) developed the Shewhart-type variance control charts based on 3-sigma approach. In this approach “3” was used as a control limits factor. Later on, numbers of authors made modification and extended the idea of 3-sigma approach to A-sigma such as Riaz et al. (2014), and Ahmad et al. (2014). Now we define the control charting structure of $V_{(G,D)}^{(E)}$ control charts based on A-sigma approach under runs rules (see, section 5.3.1).

$$UCL_{V_{(G,D)}^{(E)}} = \mu_{V_{(G,D)}^{(E)}} + A_{(k-m,k)}\sigma_{V_{(G,D)}^{(E)}},$$

$$LCL_{V_{(G,D)}^{(E)}} = \mu_{V_{(G,D)}^{(E)}} + A_{(k-m,k)}\sigma_{V_{(G,D)}^{(E)}}, \qquad (5.2)$$

where $\mu_{V_{(G,D)}^{(E)}}$ and $\sigma_{V_{(G,D)}^{(E)}}$ are means and standard deviations of the sampling distribution of $V_{(G,D),j}^{(E)}$ respectively, and A is control limits multiplier, depend on a decision rule $k-m|k$. The control limits multiplier can be derived through Monte Carlo simulation using the same procedure as mentioned in section 5.3.2.

The structures of $V_{(G,D)}^{(E)}$control charts given in Equations 5.1-5.2 represent generalized structures and have ability to accommodate some existing variance control charts as well as new proposed charts depend on the value of design parameters.

Table 5.1. Control limits factors of $\boldsymbol{V}^{(\mathbf{1})}_{(\boldsymbol{ERSS},\boldsymbol{BN})}$ control charts with different choices of $\boldsymbol{k-m}, \boldsymbol{k}, \boldsymbol{n}$ **and** $\boldsymbol{\rho_{YX}}$ at $= \mathbf{0.0027}$.

$\boldsymbol{\rho_{YX}}$	n	**Control limits factors**	$\boldsymbol{k-m\|k}$					
			1|1	**2|3**	**2|4**	**9|9**	**8|9**	**7|9**
0.30	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.7914	3.5754	3.9133	1.2296	1.4625	1.7039
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0082	0.047	0.0373	0.3812	0.2965	0.2312
	5	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.9143	3.5903	3.9439	1.2349	1.4672	1.7069
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0092	0.0474	0.0372	0.3801	0.2951	0.2302
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.3282	3.3478	3.6588	1.2409	1.4541	1.6721
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0115	0.0578	0.0456	0.4257	0.3367	0.2676
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	5.5858	2.8486	3.1008	1.2382	1.4011	1.5676
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.023	0.1033	0.0837	0.5502	0.4594	0.3829
0.50	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.7124	2.1932	2.337	1.1959	1.301	1.4053
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0581	0.2295	0.1938	0.711	0.636	0.5685
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.5585	2.5209	2.6741	1.1756	1.3373	1.4962
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0453	0.1266	0.1089	0.5065	0.4242	0.3584
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.2993	2.3872	2.523	1.1878	1.3306	1.4726
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0597	0.1595	0.1387	0.5581	0.4772	0.4095
0.75	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.7134	2.0231	2.1197	1.174	1.2815	1.3854
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.1225	0.2673	0.2402	0.6752	0.6022	0.5398
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.0829	1.66	1.715	1.1347	1.2022	1.2669
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.2842	0.4614	0.4337	0.8019	0.7486	0.7008
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.0127	2.2381	2.353	1.1715	1.3037	1.4336
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0972	0.2023	0.1809	0.5832	0.5068	0.4435
0.90	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.785	2.0998	2.2033	1.1683	1.2852	1.3976
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.1202	0.248	0.2238	0.633	0.5597	0.4973
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.2175	1.7891	1.8547	1.1473	1.2321	1.3132
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.2297	0.3811	0.3555	0.741	0.6788	0.6254
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.7562	1.4969	1.5379	1.1085	1.1611	1.2099
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.4263	0.5698	0.5479	0.844	0.8006	0.7615

Table 5.2. Control limits factors of $V^{(1)}_{(ERSS,BT)}$ control charts with different choices of $k-m, k, n$ and ρ_{YX} at $= 0.0027$.

ρ_{YX}	n	Control limits factors	$k-m\|k$					
			1|1	2|3	2|4	9|9	8|9	7|9
0.30	4	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	8.7651	3.9198	4.3506	1.2787	1.5266	1.7893
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0088	0.0462	0.0362	0.386	0.2985	0.2321
	5	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.9143	3.5903	3.9439	1.2349	1.4672	1.7069
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0092	0.0474	0.0372	0.3801	0.2951	0.2302
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.7925	3.6257	3.9777	1.2742	1.5038	1.7384
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0116	0.0596	0.0474	0.4283	0.3382	0.2686
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	6.0162	3.0002	3.2914	1.2361	1.4089	1.5855
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0222	0.1066	0.0865	0.5412	0.4521	0.3796
0.50	4	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.955	2.2599	2.4098	1.1679	1.2802	1.3911
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0586	0.2357	0.2003	0.6858	0.6153	0.5517
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	4.1126	2.8074	2.9928	1.2485	1.4267	1.6069
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0488	0.1301	0.1129	0.5231	0.4382	0.3686
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.6799	2.5752	2.7381	1.2279	1.3842	1.5397
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0594	0.1615	0.1409	0.5654	0.4837	0.4147
0.75	4	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.6361	1.9752	2.0689	1.1325	1.2371	1.338
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.1209	0.2635	0.2369	0.6504	0.5812	0.5212
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.1556	1.6767	1.7397	1.103	1.1736	1.2421
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.2771	0.4454	0.4174	0.7666	0.7152	0.6702
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.5111	2.5095	2.6542	1.2499	1.4004	1.5506
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0968	0.2055	0.1841	0.6087	0.5275	0.4598
0.90	4	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.1515	2.3014	2.4115	1.2228	1.3534	1.4822
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.1254	0.2529	0.2292	0.6466	0.571	0.5068
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.4205	1.87	1.9512	1.1486	1.238	1.3252
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.2242	0.3733	0.3483	0.7265	0.665	0.6117
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.7872	1.4865	1.5323	1.0695	1.1238	1.1747
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.4141	0.541	0.5211	0.801	0.7587	0.7216

Table 5.3. Control limits factors of $\boldsymbol{V}_{(\boldsymbol{ERSS},\boldsymbol{BLN})}^{(\mathbf{1})}$control charts with different choices of $\boldsymbol{k}-\boldsymbol{m},\boldsymbol{k},\boldsymbol{n}$ **and** $\boldsymbol{\rho}_{\boldsymbol{YX}}$at $=\mathbf{0.0027}$.

$\boldsymbol{\rho}_{\boldsymbol{YX}}$	n	**Control limits factors**	$\boldsymbol{k}-\boldsymbol{m}\vert\boldsymbol{k}$					
			1|1	**2|3**	**2|4**	**9|9**	**8|9**	**7|9**
0.30	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.9143	3.5903	3.9439	1.2349	1.4672	1.7069
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0092	0.0474	0.0372	0.3801	0.2951	0.2302
	5	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.9143	3.5903	3.9439	1.2349	1.4672	1.7069
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0092	0.0474	0.0372	0.3801	0.2951	0.2302
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.2354	3.3811	3.699	1.2404	1.4528	1.6713
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0114	0.0586	0.0462	0.4271	0.3379	0.2689
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	5.4861	2.8412	3.0867	1.2392	1.4031	1.5693
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0242	0.1058	0.0846	0.5502	0.4588	0.3828
0.50	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.6988	2.2043	2.3519	1.1943	1.2976	1.4005
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0609	0.2327	0.1979	0.7113	0.6363	0.5683
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.6034	2.5423	2.688	1.1799	1.3436	1.503
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0456	0.1268	0.1091	0.5041	0.4224	0.3562
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.3028	2.3738	2.5063	1.181	1.3245	1.4667
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0608	0.1584	0.1383	0.5554	0.4757	0.4091
0.75	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.6927	2.0341	2.1262	1.1785	1.2865	1.3908
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.1246	0.2666	0.2389	0.6765	0.604	0.5414
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.0808	1.6584	1.7143	1.1359	1.2032	1.2676
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.2838	0.4585	0.4319	0.8001	0.7459	0.6981
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.0346	2.2473	2.3637	1.1696	1.3022	1.4316
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0967	0.2006	0.1796	0.5823	0.5059	0.4435
0.90	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.771	2.0996	2.1977	1.1671	1.2851	1.3992
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.1243	0.2463	0.2237	0.6323	0.5596	0.498
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.2356	1.7865	1.8533	1.1479	1.2326	1.3139
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.2354	0.3798	0.3539	0.7395	0.6779	0.6243
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.75	1.4975	1.537	1.1093	1.1618	1.2114
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.4335	0.5706	0.5481	0.8432	0.8005	0.7616

Table 5.4. Control limits factors of $\boldsymbol{V^{(3)}_{(ERSS,BN)}}$ control charts with different choices of $\boldsymbol{k-m, k, n}$ **and** $\boldsymbol{\rho_{YX}}$ at $= \mathbf{0.0027}$.

ρ_{YX}	n	Control limits factors	$k-m\|k$					
			1|1	**2|3**	**2|4**	**9|9**	**8|9**	**7|9**
0.30	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	9.8618	3.8598	4.4267	0.9469	1.1777	1.4287
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2144	0.15	0.1031
	5	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	9.8618	3.8598	4.4267	0.9469	1.1777	1.4287
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2144	0.15	0.1031
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	9.0343	3.6177	4.128	0.9023	1.1198	1.3631
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1941	0.129	0.0795
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	6.7821	3.1308	3.5163	0.8504	1.0442	1.2503
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1705	0.0926	0.0187
0.50	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	4.441	2.4211	2.6588	0.8401	0.996	1.1584
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1662	0.0377	0
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.1179	1.9234	2.0862	0.7604	0.8861	1.012
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2683	0.2127	0.1684
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.0162	1.8344	1.9812	0.7007	0.8204	0.9423
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2269	0.1696	0.1215
0.75	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.6412	1.6205	1.7532	0.6103	0.7192	0.8299
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1343	0.0539	0
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.1272	1.403	1.4974	0.5799	0.6795	0.7746
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.0233	0	0
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.0946	1.4632	1.5506	0.6806	0.7729	0.8647
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0.0145	0	0.288	0.2383	0.1969
0.90	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.9269	1.3238	1.4078	0.591	0.6769	0.7615
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2179	0.1667	0.1215
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.6328	1.0955	1.1689	0.453	0.5309	0.6073
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.0555	0	0
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.3695	0.9604	1.0204	0.3808	0.4612	0.5334
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0	0	0

Table 5.5. Control limits factors of $V^{(3)}_{(ERSS,BT)}$ control charts with different choices of $k-m, k, n$ **and** ρ_{YX} at $= 0.0027$.

ρ_{YX}	n	Control limits factors	$k-m\|k$					
			1|1	**2|3**	**2|4**	**9|9**	**8|9**	**7|9**
0.30	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	10.8557	4.1635	4.7654	0.944	1.1873	1.455
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1909	0.1272	0.0807
	5	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.9143	3.5903	3.9439	1.2349	1.4672	1.7069
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0092	0.0474	0.0372	0.3801	0.2951	0.2302
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	9.4947	3.8335	4.4059	0.8934	1.1213	1.3733
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1642	0.0984	0.0475
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.2817	3.2471	3.6456	0.8176	1.0198	1.2369
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1119	0.0205	0
0.50	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	4.7372	2.4723	2.7182	0.8011	0.9667	1.1352
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.0582	0	0
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.3747	2.0362	2.2009	0.7614	0.8942	1.0325
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2479	0.1908	0.1436
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.1902	1.8829	2.0544	0.679	0.8039	0.9311
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1844	0.122	0.0666
0.75	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.6026	1.5923	1.7259	0.5846	0.6943	0.8047
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.0999	0.0131	0
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.1119	1.3722	1.4775	0.5091	0.6152	0.7182
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0	0	0
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.2623	1.5486	1.6466	0.6811	0.7789	0.878
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2672	0.2146	0.171
0.90	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.0336	1.3516	1.4476	0.567	0.6594	0.7505
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1686	0.1098	0.0534
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.6898	1.074	1.1486	0.3867	0.4701	0.5515
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0	0	0
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.3375	0.9029	0.9692	0.2779	0.368	0.45
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0	0	0

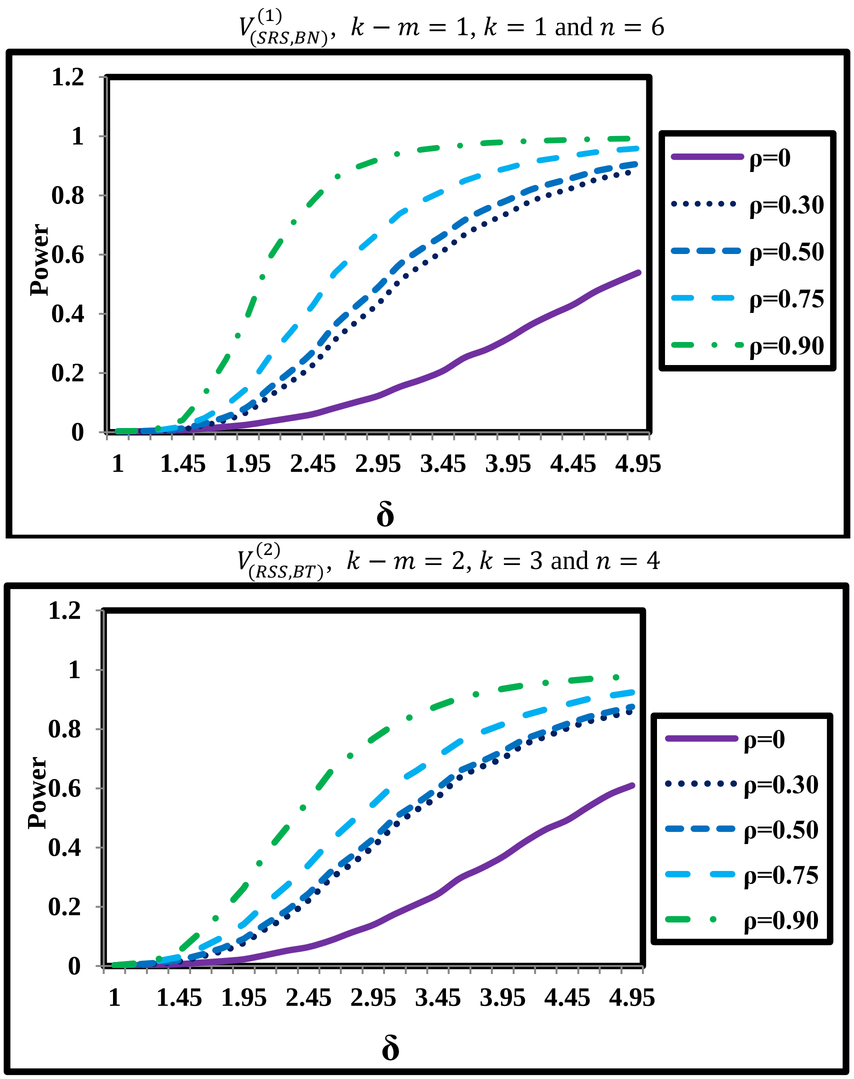
Table 5.6. Control limits factors of $\boldsymbol{V}^{(2)}_{(ERSS,BLN)}$control charts with different choices of $\boldsymbol{k-m, k, n}$ **and** $\boldsymbol{\rho_{YX}}$at $= \mathbf{0.0027}$.

$\boldsymbol{\rho_{YX}}$	n	**Control limits factors**	$\boldsymbol{k-m\|k}$					
			1|1	**2|3**	**2|4**	**9|9**	**8|9**	**7|9**
0.30	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	9.9547	3.9157	4.4743	0.9518	1.1827	1.4338
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2127	0.1495	0.1031
	5	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	7.9143	3.5903	3.9439	1.2349	1.4672	1.7069
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0.0092	0.0474	0.0372	0.3801	0.2951	0.2302
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	9.0166	3.6902	4.187	0.906	1.1269	1.3653
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1953	0.1303	0.0803
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	6.7333	3.1297	3.4704	0.8508	1.0434	1.2554
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1706	0.0916	0.017
0.50	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	4.4405	2.4154	2.6511	0.8406	0.996	1.157
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1655	0.0374	0
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	3.1614	1.9381	2.0955	0.7618	0.8889	1.0199
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2677	0.2122	0.1675
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.99	1.8333	1.9784	0.6983	0.8163	0.937
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2253	0.1677	0.1194
0.75	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.5848	1.6126	1.7459	0.6116	0.7215	0.8315
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.1355	0.0561	0
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.0995	1.3891	1.4811	0.5806	0.6799	0.7753
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.0249	0	0
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	2.1047	1.4637	1.5597	0.6788	0.7714	0.8624
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0.0164	0	0.2876	0.2387	0.1981
0.90	**4**	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.9449	1.3232	1.4133	0.5905	0.675	0.7582
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.2186	0.1676	0.1212
	6	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.6237	1.0933	1.166	0.4526	0.531	0.6064
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0.0558	0	0
	8	$L_{(E,G,D,k-m,k,n,\rho_{YX}1-,\frac{p}{2})}$	1.3699	0.9587	1.0202	0.3813	0.4598	0.5329
		$L_{(E,G,D,k-m,k,n,\rho_{YX},\frac{p}{2})}$	0	0	0	0	0	0

5.4 Performance evaluation and comparisons

In this section, we investigate the performance of $V_{(G,D)}^{(E)}$ control charts structures (see, section 3) in term of power curve. The literature in favor of assessing performance of a control chart through power curve is as follows: Riaz et al. (2011), Mehmood et al. (2013b) and Mehmood et al. (2014). We define the power curve as the graphical representation of the probability of at least $k-m$ values of the test statistic $V_{(G,D),j}^{(E)}$ out of k consecutive values either exceed the $LCL_{V_{(G,D)}^{(E)}}$ or $UCL_{V_{(G,D)}^{(E)}}$ when a certain amount of shift δ occurs in the process parameter of the variable of interest $\sigma_{Y(D)}^{2}$. The shift in the process parameter of the variable of interest Y is defined as: $\sigma_{Y(D)(\delta)}^{2}=(\delta\sigma_{Y(D)})^{2}$. Moreover, $\delta=1$ implies that no shift occurs in the process parameter and the process behaving naturally, whereas, $\delta>1$ means that a special cause of variation is interrupting the process and switching it from in-control state to out-of-control. There are several studies in which shifts have considered in the dispersion parameter of the variable of interest Y while dealing the auxiliary information based dispersion control charts such as Riaz et al. (2014), and Ahmad et al. (2014). For the computation of power curve, we assume that process follows bivariate distribution D with $\mu_Y=0$, $\mu_X=0$, $\sigma_Y^2=1$, $\sigma_X^2=1$, and $\upsilon=30$. Afterwards, for a given value of design parameters $E, G, D, k-m, k, n$ and ρ_{YX} at $\alpha=0.0027$, pick the required control limits factors given in Tables 5.1-5.6, and construct the control limits (based on probability limits approach). In the next step, for a given value of design parameters and δ, generates the random samples from bivariate distribution D (such as BN, BT and BLN) and check whether the value of test statistic $V_{(G,D),j}^{(E)}$ is inside or outside the control limits ($LCL_{V_{(G,D)}^{(E)}}$, $UCL_{V_{(G,D)}^{(E)}}$). The following procedure is repeated 10^5 for varying values of the design parameters with different amount of δ and finally, proportions of the test statistic $V_{(G,D),j}^{(E)}$ beyond the control limits ($LCL_{V_{(G,D)}^{(E)}}$, $UCL_{V_{(G,D)}^{(E)}}$) are displayed in the form of Figure 5.1-5.5.

Figure 5. 1 Power curves of $\boldsymbol{V_{(G,D)}^{(E)}}$control charts for a given value of $\boldsymbol{E, G, D, k-m, k}$ **and** $\boldsymbol{n}$ with an increase in $\boldsymbol{\rho_{YX}}$ at $\boldsymbol{\alpha = 0.0027}$



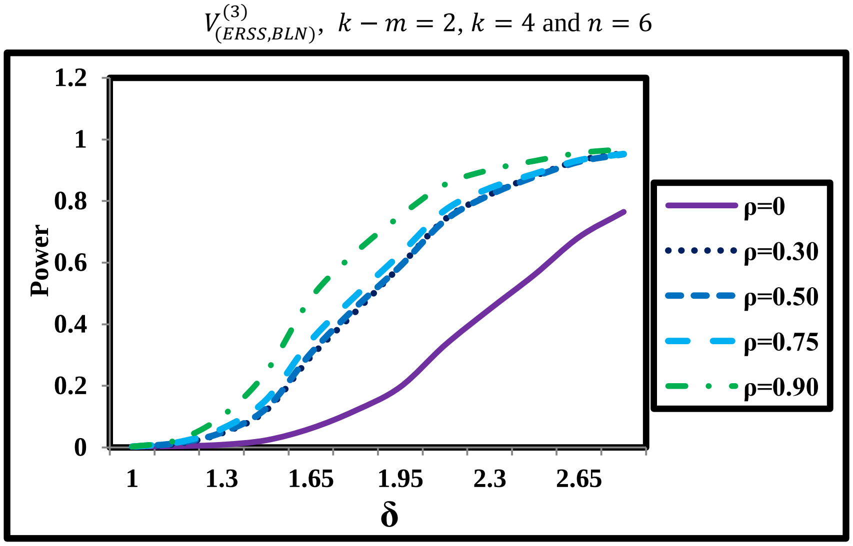
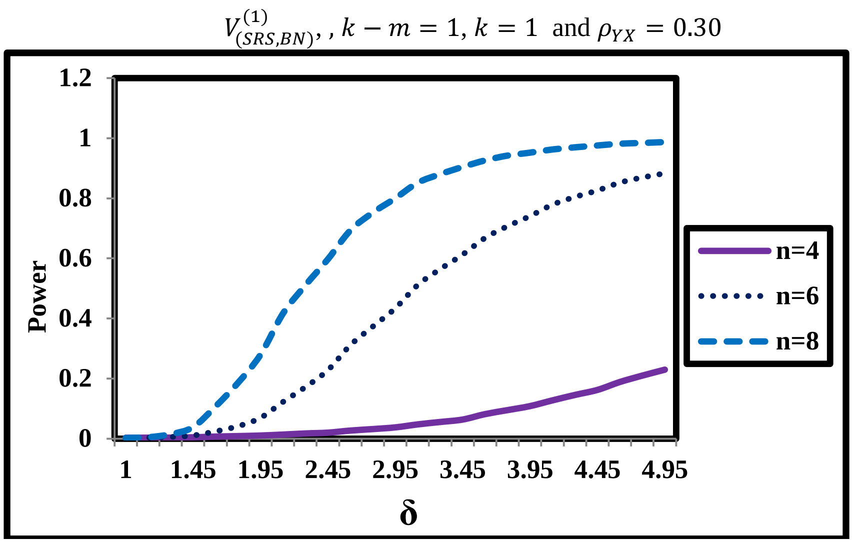
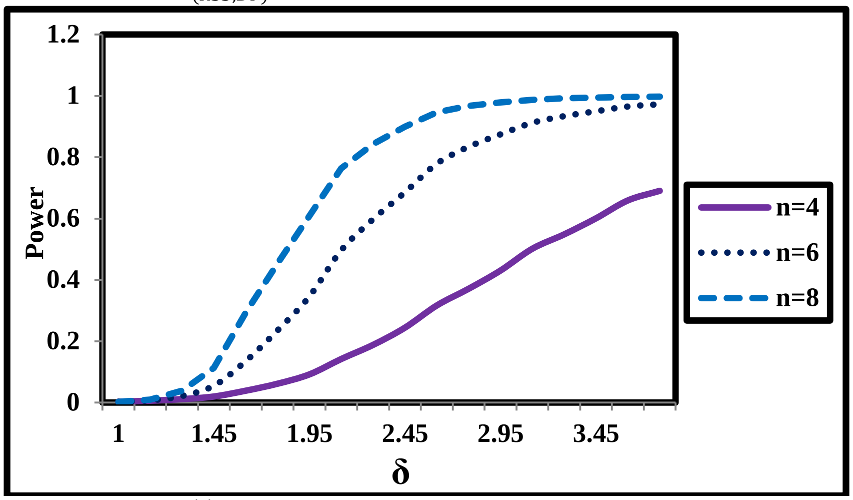


Figure 5.2. Power curves of $\boldsymbol{V^{(E)}_{(G,D)}}$ control charts for a given value of $\boldsymbol{E, G, D, k-m, k}$ **and** $\boldsymbol{\rho_{YX}}$ with different sample sizes $\boldsymbol{n}$ at $\boldsymbol{\alpha = 0.0027}$.



$V^{(2)}_{(RSS,BT)}$, $k - m = 2, k = 3$ and $\rho_{YX} = 0.50$



$V^{(3)}_{(ERSS,BLN)}$, $k - m = 2, k = 4$ and $\rho_{YX} = 0.75$

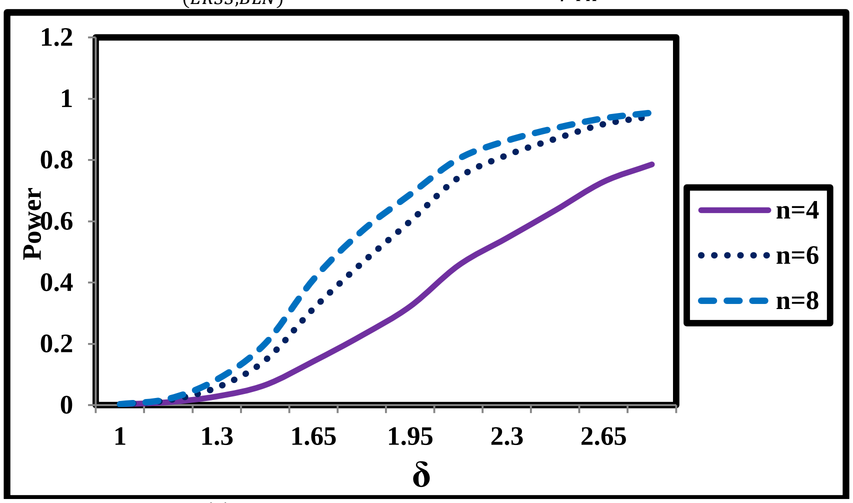
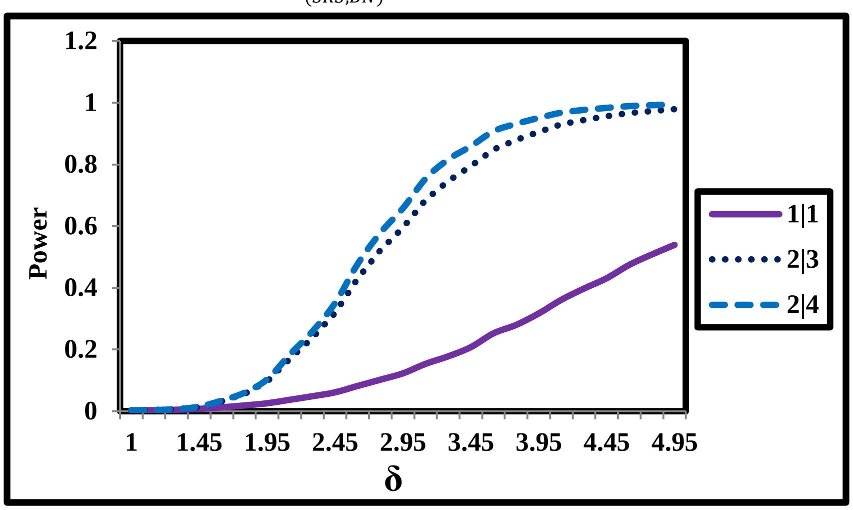
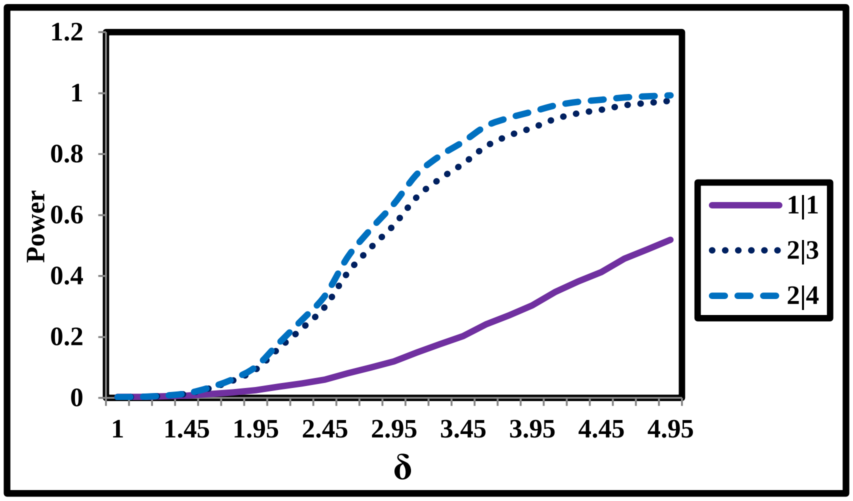


Figure 5.3.Power curves of of $\boldsymbol{V^{(E)}_{(G,D)}}$control charts for a given value of $\boldsymbol{E, G, D, n}$ **and** $\boldsymbol{\rho_{YX}}$ with different choices of $\boldsymbol{k - m}$ and $\boldsymbol{k}$at $\boldsymbol{\alpha = 0.0027}$.

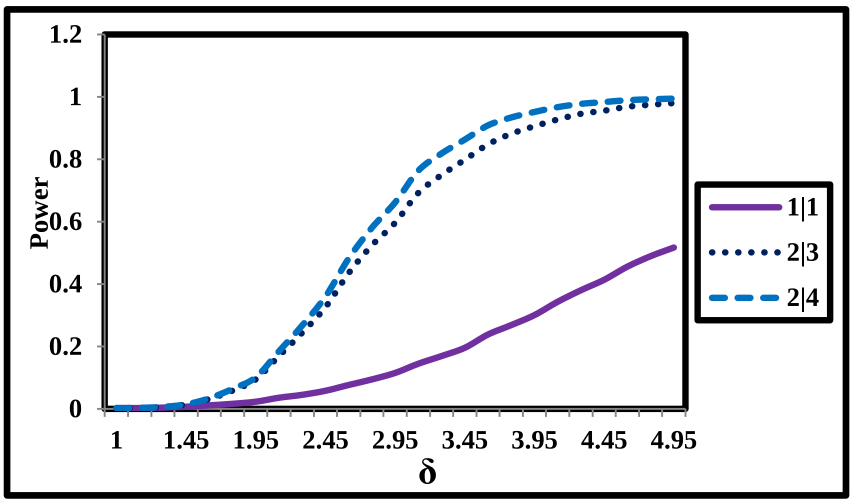
$V^{(1)}_{(SRS,BN)}$, $\rho_{YX} = 0$ and $n = 6$



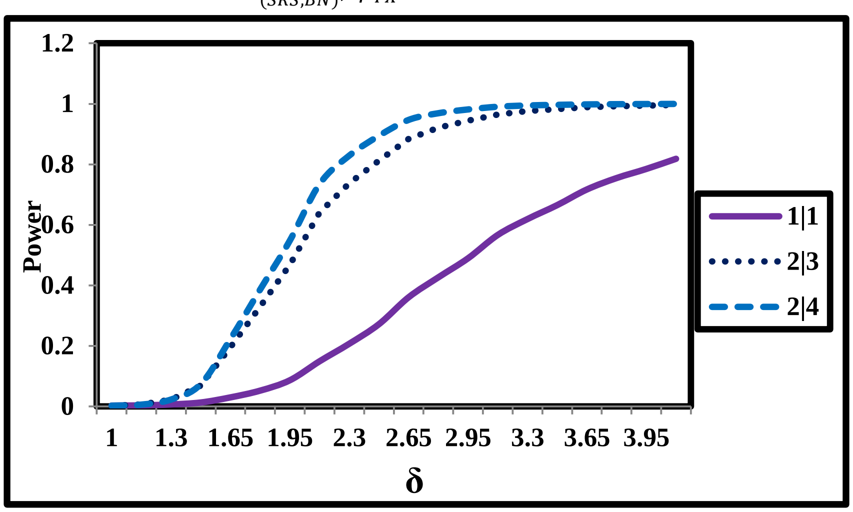
$V^{(2)}_{(SRS,BT)}$, $\rho_{YX} = 0$ and $n = 6$



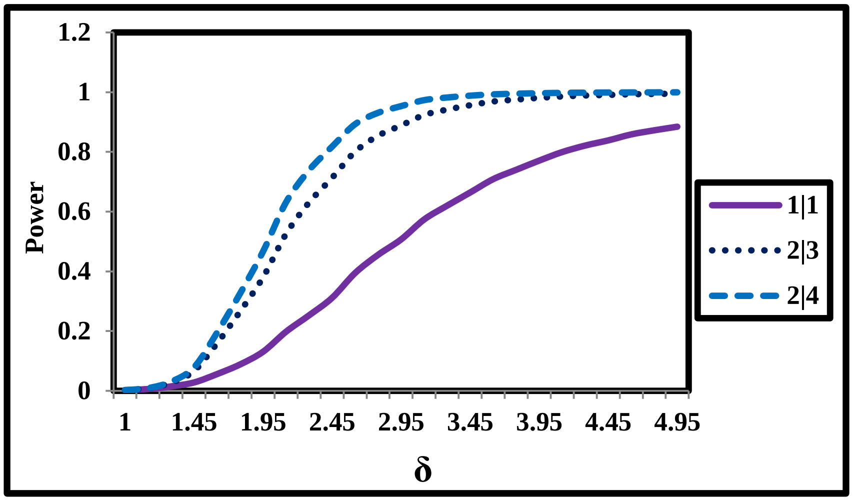
$V^{(2)}_{(SRS,BLN)}$, $\rho_{YX} = 0.0$ and $n = 8$



$V^{(1)}_{(SRS,BN)}$, $\rho_{YX} = 0.50$ and $n = 6$



$V^{(2)}_{(RSS,BT)}$, $\rho_{YX} = 0.75$ and $n = 6$



$V^{(2)}_{(ERSS,BLN)}$, $\rho_{YX} = 0.30$ and $n = 8$

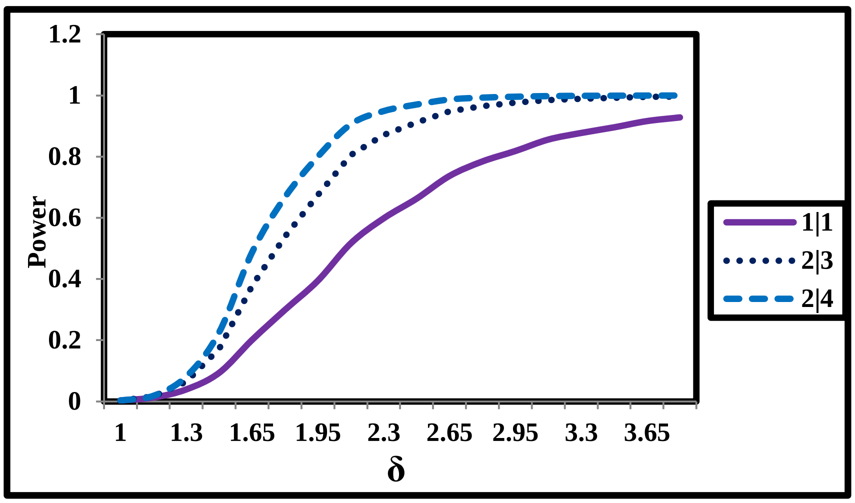
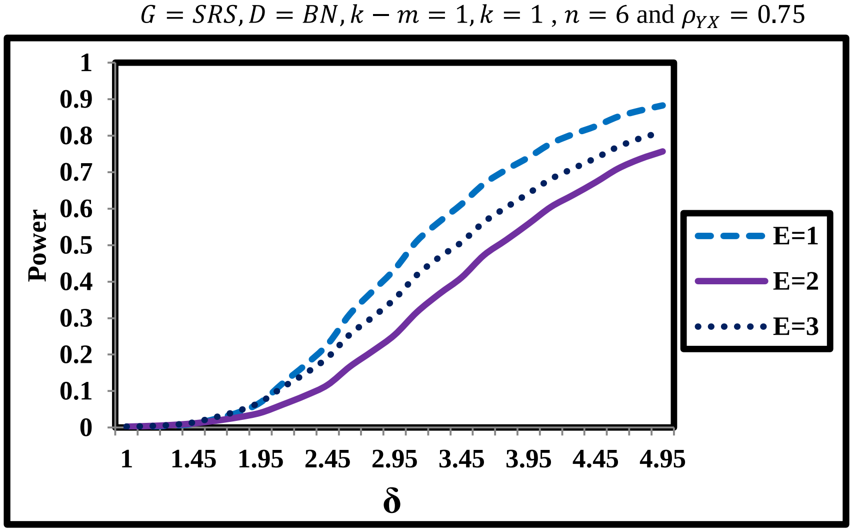
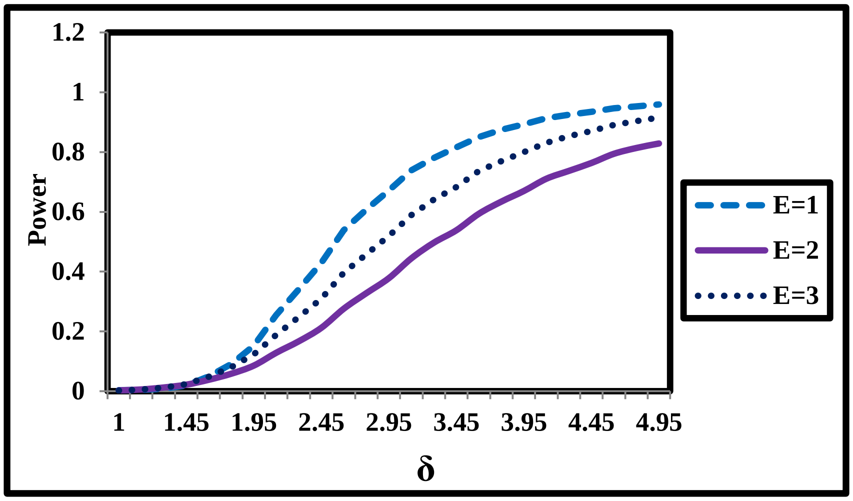


Figure 5.4. Power curves of of $\boldsymbol{V^{(E)}_{(G,D)}}$control charts for a given value of $\boldsymbol{G, D, k-m, k, n}$ **and** $\boldsymbol{\rho_{YX}}$ with different choices $\boldsymbol{E}$ at $\boldsymbol{\alpha = 0.0027}$.

$G = SRS, D = BN, k - m = 1, k = 1$, $n = 6$ and $\rho_{YX} = 0.75$



$G = RSS, D = BN, k - m = 1, k = 1$, $n = 6$ and $\rho_{YX} = 0.75$



$G = ERSS, D = BT, k - m = 2, k = 3$, $n = 6$ and $\rho_{YX} = 0.75$

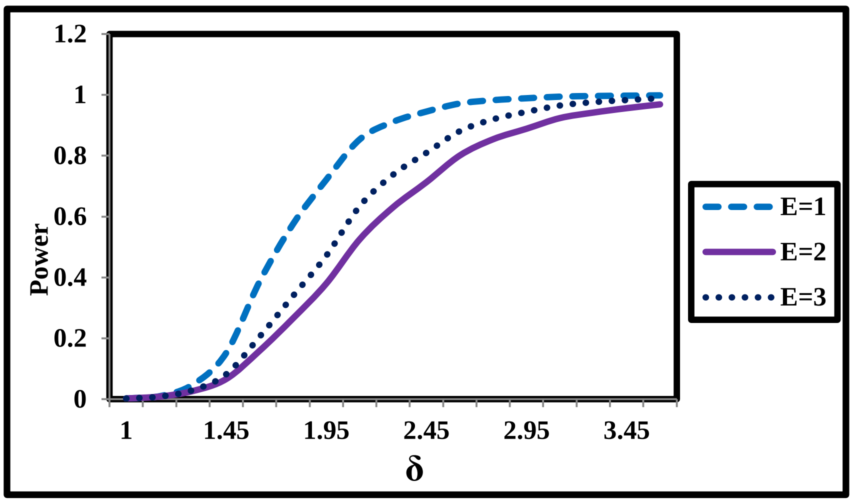
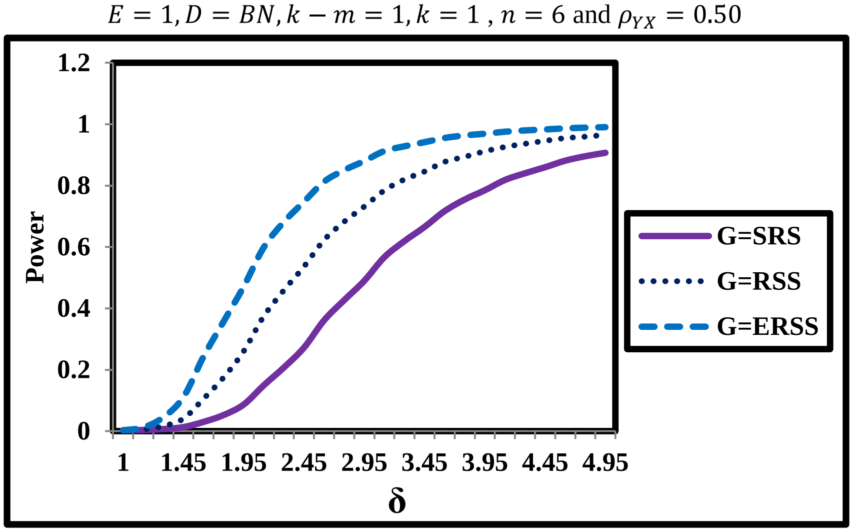
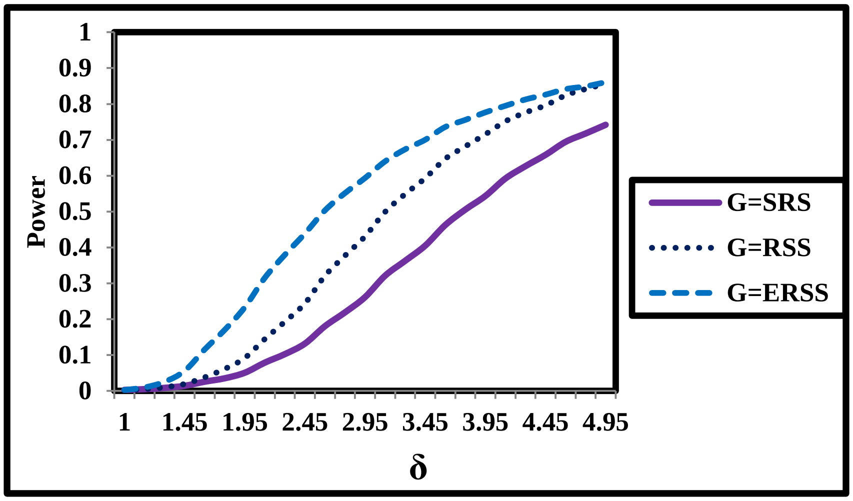


Figure 5.5. Power curves of of $\boldsymbol{V}_{(G,D)}^{(E)}$ control charts for a given value of $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{k-m}, \boldsymbol{k}, \boldsymbol{n}$ **and** $\boldsymbol{\rho_{YX}}$ with different choices $\boldsymbol{G}$ at $\boldsymbol{\alpha = 0.0027}$.

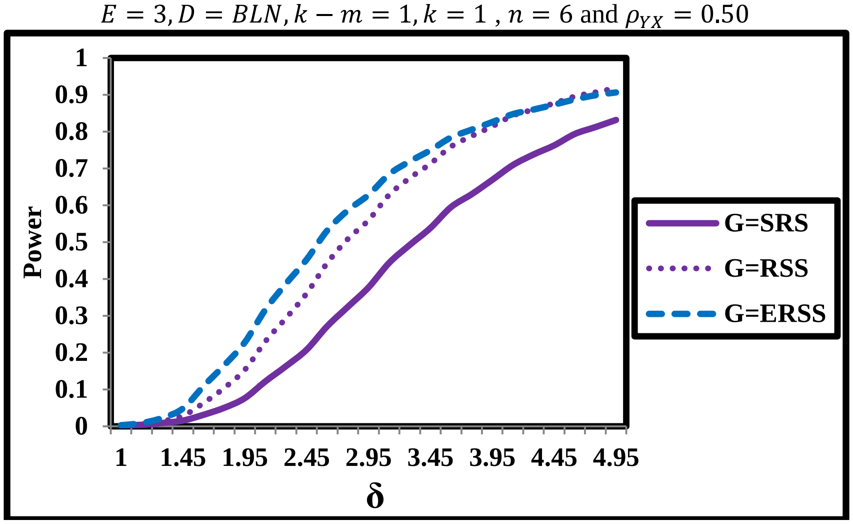
$E = 1, D = BN, k - m = 1, k = 1$, $n = 6$ and $\rho_{YX} = 0.50$



$E = 2, D = BT, k - m = 1, k = 1$, $n = 6$ and $\rho_{YX} = 0.50$



$E = 3, D = BLN, k - m = 1, k = 1$, $n = 6$ and $\rho_{YX} = 0.50$



5.5 Results and Discussion

In this section we briefly illustrate the results and also give discussion based on Figures 5.1-5.5. Firstly, motivation of the following section is to evaluate the proposed structure, in order to check whether it has ability to keep on statistical properties which is generally considered important for designing an efficient structure, Secondly, is it according to the subjects of the current study and can be treated as a generalized form of several variance control charts. Here, generalization implies that to facilitate the practitioners by providing a generalized structure at single place, which not only cover dual auxiliary information control charts, but also provide an improved version of existing control charts. The results and discussion are given in the following points:

- The performance of $V_{(G,D)}^{(E)}$ control charts is monotonic-increasing in $k-m, k, n, \rho_{YX}$ and δ. An increase in any of these quantities substantially increases the detection ability of the charts (see, Figures 5.1-5.3) in general. The effect of $k-m, k, n$ and ρ_{YX} on the performance of control charts is also mentioned by Mehmood et al. (2013b) and Mehmood et al. (2014).
- Figure 3 depicted that rule 1/1 is efficient for the detection of larger shifts, whereas other runs rules boost the performance of $V_{(G,D)}^{(E)}$ control charts for smaller and moderate shifts. So, the following result has fulfilled the recommendations of the study as mentioned in section 5.1. Also, behavior of different runs rules is in accordance with Riaz et al. (2011).
- For a given sampling strategy G and bivariate distribution$D$, $V_{(G,D)}^{(1)}$ control charts exhibited best performance followed by $V_{(G,D)}^{(3)}$and $V_{(G,D)}^{(2)}$ in general for all choices

of $k-m, k, n$ and ρ_{YX} . So, control charts based on $E=1$ observed at 1st position followed by control charts based on $E=3$ and $E=2$ in general (see, Figure 5.4).

- Among different sampling strategy control charts based on ERSS are declared at first position followed by control charts based on RSS and SRS in general (see, Figure 5.5)
- It is clear from Figure 5.5, performance of dual auxiliary information based control charts (such as $V_{(RSS,D)}^{(E)}$, and $V_{(ERSS,D)}^{(E)}$) is uniformly better than simple random sampling based counterpart, such as Riaz et al. (2014). This shows that use of auxiliary information at dual stages boosts the performance of control charts.

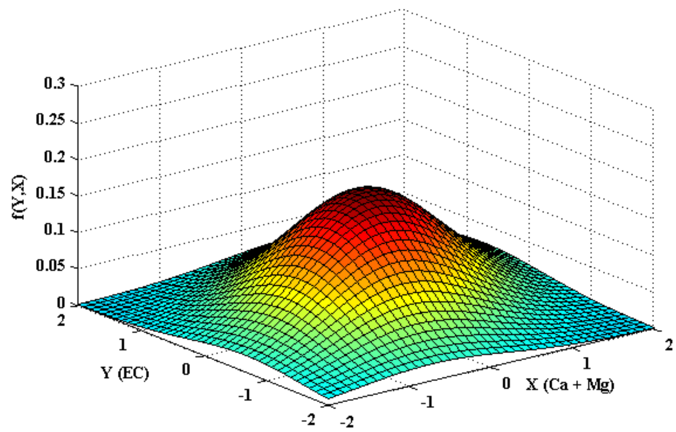
5.5.1 Special cases

Control charting structures given in Equations (5.1)-(5.2) are generalized structures which have ability to accommodate other existing control charting structure (see, Figure 5.3). For instance, Shewhart (1931) when $E=1, G=SRS, D=BN, k-m|k=1|1$ and $\rho_{YX}=0$; Riaz (2008b) when $E=1, G=SRS, D=BN, k-m|k=1|1$; Riaz et al. (2011) when $E=1, G=SRS, D=BN, k-m|k=(1|1,2|3,2|4)$ and $\rho_{YX}=0$; Riaz et al. (2014) when $E=(1,2,3), G=SRS, D=BN, k-m|\ k=1|1$. Moreover, control charts proposed by Shewhart (1931), Riaz (2008b), Riaz et al. (2011), and Riaz et al. (2014) are based on the normality assumption and one point decision rule. We have extended and improved their design structures by incorporating the other bivariate distribution, different sampling strategies and runs rules (see, Figures 5.3 & 5.5).

5.6 A real life example of $V_{(G,D)}^{(E)}$ control charts

In this section we provide a real life application of $V_{(G,D)}^{(E)}$control charts for monitoring the stability of physico-chemical parameters of groundwater. Monitoring of these parameters over time based on the groundwater sample has been considered as vital factors for a number of reasons such as production of crop yield, and drinking water for human health A crop yield depends on several physico-chemical parameters of groundwater such as color, temperature, acidity, hardness, PH and sulphite. Monitoring of each parameter is considered important for good crops yield. Among these parameters, we consider two physico-chemical parameters of groundwater to show the application of proposed charts, which include total dissolved solids and total hardness of water. Total dissolved solids (measured in term of electric conductivity (EC)) is considered as study variable Y, whereas, total hardness of water (measured in term of calcium magnesium carbonates) is taken as an auxiliary variable. We consider groundwater (used for irrigation of crop yield) of District Rahim Yar Khan, Pakistan as a case study in which water samples of size 5 are collected from thirty different locations using extreme ranked set sampling and on which we perform actual measurement. The actual measurements of electric conductivity and calcium magnesium carbonates are tabulated in the form of Table 5.7. We also draw probability density plot of the bivariate variables under consideration (see, Figure 5.6).

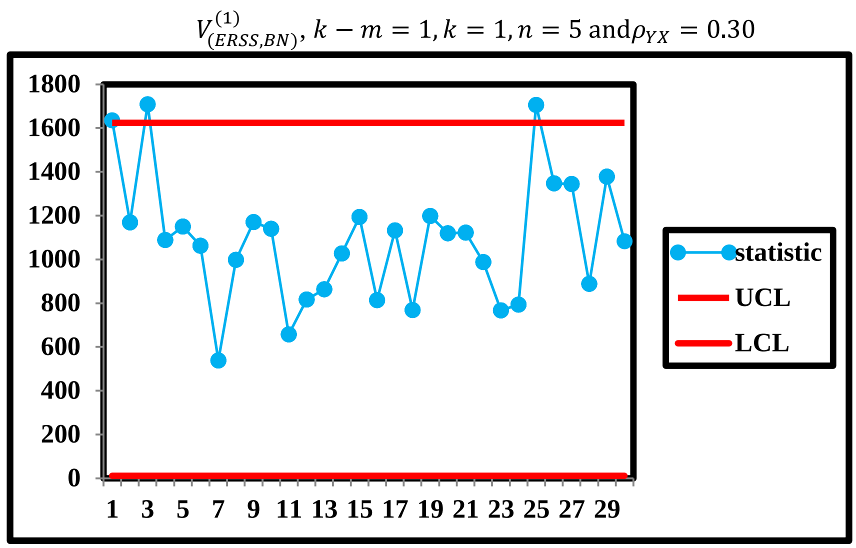
Figure 5.6. probability density plot of the bivariate variables (Electric conductivity and calcium magnesium carbonates)

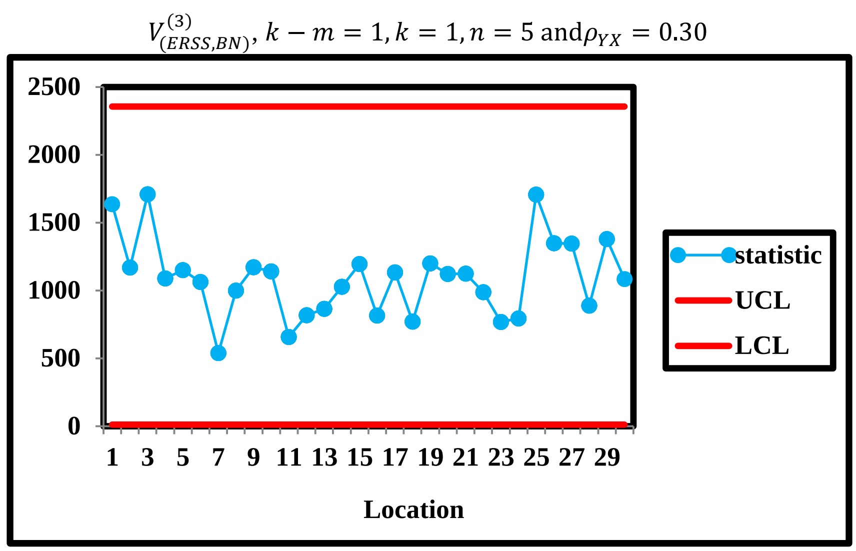


Before implementing the control charts we calculated two statistics $V^{(1)}_{(ERSS,BN),j}$ and $V^{(3)}_{(ERSS,BN),j}$ using the data set given in Table 5.7 with known in-control parameters are $\mu_Y = 836.06, \mu_X = 4.93, \sigma^2_Y = 400,$ and $\sigma^2_X = 1.53$

We implemented two control charts ($V^{(1)}_{(ERSS,BN)}$ and $V^{(3)}_{(ERSS,BN)}$) based on probability limits approach for monitoring the variability of each water sample of a given location with respect to electric conductivity (see, Figure 5.7).

Figure 5.7. Monitoring the dispersion parameter of electric conductivity through control charts





From Figure 7, it is clear that electric conductivity of three locations is out-of-control, detected by $V_{(ERSS,BN)}^{(1)}$ control chart, whereas $V_{(ERSS,BN)}^{(3)}$ control chart did not show any out-of-control signal. Although, both control charts have ability to monitor the variation in the dispersion parameter (electric conductivity), but the performance of $V_{(ERSS,BN)}^{(1)}$ is superb compared to $V^{(3)}_{(ERSS,BN)}$. The following outcomes are in accordance with the results and discussion of the current study (see section 4).

Table 5.7. Actual measurement of electric conductivity *Y* and calcium-magnesium carbonates *X*

Location (*j*)		Observation (*i*)				
		1	2	3	4	5
1	$Y_{[i,erss],1}$	860	817	880	965	856
	$X_{(i,erss),1}$	6.4	5.8	4.4	6.5	4.8
2	$Y_{[i,erss],2}$	846	830	845	890	897
	$X_{(i,erss),2}$	3.8	3.5	3.6	3.8	4.2
3	$Y_{[i,erss],3}$	850	828	879	803	887
	$X_{(i,erss),3}$	7.5	6.3	6	6.2	6.9
4	$Y_{[i,erss],4}$	806	835	780	790	757
	$X_{(i,erss),4}$	4.7	2.7	3	3.7	5.1
5	$Y_{[i,erss],5}$	750	792	760	720	791
	$X_{(i,erss),5}$	4.2	5.8	4.6	3.7	3
6	$Y_{[i,erss],6}$	744	720	790	775	782
	$X_{(i,erss),6}$	3.2	3.6	3.5	4.9	4.7
7	$Y_{[i,erss],7}$	895	830	877	869	844
	$X_{(i,erss),7}$	1.9	1.8	1.7	2	3
8	$Y_{[i,erss],8}$	888	825	860	880	895
	$X_{(i,erss),8}$	4.5	5.1	4.8	4.7	5
9	$Y_{[i,erss],9}$	775	792	750	812	742
	$X_{(i,erss),9}$	5	4	4.9	3	3.7
10	$Y_{[i,erss],10}$	868	850	885	890	820
	$X_{(i,erss),10}$	3.5	3.8	4	4.5	3.2
11	$Y_{[i,erss],11}$	873	850	834	890	849
	$X_{(i,erss),11}$	2.5	1.8	1.7	1.8	2
12	$Y_{[i,erss],12}$	792	811	816	845	870
	$X_{(i,erss),12}$	5.8	5.9	4.7	5.3	6
13	$Y_{[i,erss],13}$	933	870	909	933	925
	$X_{(i,erss),13}$	6	6.5	6	6.3	6.6
14	$Y_{[i,erss],14}$	910	909	888	933	860
	$X_{(i,erss),14}$	5.3	7.2	7.3	6.3	6.2
15	$Y_{[i,erss],15}$	912	960	925	888	890
	$X_{(i,erss),15}$	7	6.3	6.6	7.3	5.1
16	$Y_{[i,erss],16}$	867	860	814	795	835
	$X_{(i,erss),16}$	5	6	4.5	6.5	6.4
17	$Y_{[i,erss],17}$	920	909	867	890	945
	$X_{(i,erss),17}$	6.4	6.2	5.2	5.9	7.2
18	$Y_{[i,erss],18}$	740	792	732	743	743
	$X_{(i,erss),18}$	4.8	5.8	4.3	4.2	5.3
19	$Y_{[i,erss],19}$	760	725	790	795	750
	$X_{(i,erss),19}$	5.7	5.5	5.9	5.8	5.6
20	$Y_{[i,erss],20}$	781	740	798	803	812
	$X_{(i,erss),20}$	4	3.6	5	5	5.2
21	$Y_{[i,erss],21}$	870	845	835	828	773
	$X_{(i,erss),21}$	6.8	6.3	6.4	6.2	6
22	$Y_{[i,erss],22}$	858	873	830	880	820
	$X_{(i,erss),22}$	3.1	3.3	3	3.5	3.7
23	$Y_{[i,erss],23}$	745	773	730	780	732
	$X_{(i,erss),23}$	3.8	4.8	3.6	3.9	4.3
24	$Y_{[i,erss],24}$	937	933	909	874	933
	$X_{(i,erss),24}$	6.7	6	4.7	4	6.3
25	$Y_{[i,erss],25}$	872	903	914	830	856
	$X_{(i,erss),25}$	7	6.7	7.3	6.4	6
26	$Y_{[i,erss],26}$	830	780	867	820	825
	$X_{(i,erss),26}$	5.7	5.5	5.9	5.7	5.6
27	$Y_{[i,erss],27}$	750	740	860	860	768
	$X_{(i,erss),27}$	5.8	4.8	5.5	5.3	5
28	$Y_{[i,erss],28}$	880	840	867	909	867
	$X_{(i,erss),28}$	5	4.3	4.9	5.2	4.8
29	$Y_{[i,erss],29}$	880	918	915	890	840
	$X_{(i,erss),29}$	6.3	6.8	6.6	6.8	6
30	$Y_{[i,erss],30}$	747	730	730	720	790
	$X_{(i,erss),30}$	2.9	3.6	2.2	3.6	3.2

5.7 Summary and conclusions

In this chapter, we have proposed a set of variance control charts $V_{(G,D)}^{(E)}$ for the monitoring of dispersion parameter. We have investigated the performance of different charting structures through power curve and concluded that control charts based on dual use of auxiliary information ($V_{(RSS,D)}^{(E)}$ and $V_{(ERSS,D)}^{(E)}$) have superior performance as compared to single use of auxiliary information based control charts ($V_{(SRS,D)}^{(E)}$). Among different sampling strategies, the control charts based on ERSS are rated first, followed by control charts based on RSS and SRS in general. For a given sampling strategy G and bivariate distribution$D$, $V_{(G,D)}^{(1)}$ control charts exhibited best performance followed by $V_{(G,D)}^{(3)}$and $V_{(G,D)}^{(2)}$ for all choices of $k-m, k, n$ and ρ_{YX}. Moreover, the proposed structure is a generalized form that accommodates the other existing counterparts. Our study also facilitates the practitioner to understand the procedural details of different estimators and control charts through a real life example.

CHAPTER 6

SUMMARY, CONCLUSIONS AND FUTURE RECOMMENDATIONS

6.1 Summary and conclusions

In this thesis we developed generalized structures of Shewhart control charts for the monitoring of location and dispersion parameters. In order to develop these structure we used robust dispersion estimators, auxiliary information based location and dispersion estimators, and an efficient use of Cornish fisher expansion in the form of skewness correction method. For performance evaluation we considered false alarm rate and probability to signals as performance measures. The outcomes of the thesis showed that the new design structures of Shewhart control charts perform outstandingly compared to their counterparts. Also, new designed schemes can be treated as generalized form of some existing schemes. Moreover, practical applications and numerical illustrations are also included to verify the study for practical purposes.

6.2 Future recommendations

Our future recommendations are given in the following points:

- Extending the work in the direction of exponentially weighted moving average (EWMA) control chart and cumulative sum control chart (CUSUM).
- Improving the design structure of mixed Shewhart-EWMA and Shewhart-CUSUM control charts.

Appendices

Appendix A

Cornish fisher expansion in the context of runs rules

Let X be the standardized random variable with mean zero and standard deviation one, $X_{(k,m,p)}$ and $Z_{(k,m,p)}$ be the pth-quantile of X and the standard normal distribution for a given value of k and m, respectively, and k_r be the rth cummulant of X ($r \geq 3$). Then $X_{(k,m,p)}$ has the Cornish fisher expansion defined under the run rules (see Chan & Cui, 2003; and Riaz et al., 2011) as follows:

$$X_{(k,m,p)} = Z_{(k,m,p)} + \frac{1}{6}\left(Z^2_{(k,m,p)} - 1\right)k_3 + \frac{1}{24}\left(Z^3_{(k,m,p)} - 3Z_{(k,m,p)}\right)k_4 - \frac{1}{36}\left(2Z^3_{(k,m,p)} - 5Z_{(k,m,p)}\right)k_3 + \cdots$$

In the above expression pth-quantile is divided into $(\frac{p}{2})th$ and $(1-\frac{p}{2})th$ quantile and written as:

$$X_{(k,m,\frac{p}{2})} = Z_{(k,m,\frac{p}{2})} + \frac{1}{6}\left(Z^2_{(k,m,\frac{p}{2})} - 1\right)k_3 + \frac{1}{24}\left(Z^3_{(k,m,,\frac{p}{2})} - 3Z_{(k,m,,\frac{p}{2})}\right)k_4 - \frac{1}{36}\left(2Z^3_{(k,m,\frac{p}{2})} - 5Z_{(k,m,\frac{p}{2})}\right)k_3 + \ldots\ldots \tag{A1}$$

$$X_{(k,m,1-\frac{p}{2})} = Z_{(k,m,1-\frac{p}{2})} + \frac{1}{6}\left(Z^2_{(k,m,1-\frac{p}{2})} - 1\right)k_3 + \frac{1}{24}\left(Z^3_{(k,m,1-\frac{p}{2})} - 3K_{(k,m,1-\frac{p}{2})}\right)k_4 - \frac{1}{36}\left(2Z^3_{(k,m,1-\frac{p}{2})} - 5Z_{(k,m,1-\frac{p}{2})}\right)k_3 + \ldots\ldots \tag{A2}$$

Now, similar to the Chan & Cui (2003), approximate of (A1) and (A2) are given as:

$$X_{(k,m,\frac{p}{2})} = Z_{(k,m,\frac{p}{2})} + \frac{1}{6}\left(Z^2_{(k,m,\frac{p}{2})} - 1\right)k_3 h(k_3)\ ,\ X_{(k,m,1-\frac{p}{2})} = Z_{(k,m,1-\frac{p}{2})} + \frac{1}{6}\left(Z^2_{(k,m,1-\frac{p}{2})} - 1\right)h(k_3),$$

where, $h(k_3) = \frac{1}{1+0.2k_3^2}$ is the smooth function, such that, $h(0) = 1$, $h(k_3) = h(-k_3)$ and $\lim_{|k_3|\to\infty} k_3\, h(k_3) = 0$. Finally, the quantiles ($X_{(k,m,\frac{p}{2})}$, $X_{(k,m,1-\frac{p}{2})}$) of different control charts under consideration are given below:

$\bar{X}_{RRSC}$ Chart: $\quad Z_{(k,m,\frac{p}{2})} + \frac{\frac{1}{6}\left(Z^2_{(k,m,\frac{p}{2})}-1\right)k_3(\bar{X})}{1+0.2k_3^2(\bar{X})}\quad , \quad Z_{(k,m,1-\frac{p}{2})} + \frac{\frac{1}{6}\left(Z^2_{(k,m,1-\frac{p}{2})}-1\right)k_3(\bar{X})}{1+0.2k_3^2(\bar{X})}$

R_{RRSC} Chart: $\quad Z_{(k,m,\frac{p}{2})} + \frac{\frac{1}{6}\left(Z^2_{(k,m,\frac{p}{2})}-1\right)k_3(R)}{1+0.2k_3^2(R)}\quad , \quad Z_{(k,m,1-\frac{p}{2})} + \frac{\frac{1}{6}\left(Z^2_{(k,m,1-\frac{p}{2})}-1\right)k_3(R)}{1+0.2k_3^2(R)}$

S_{RRSC} Chart: $\quad Z_{(k,m,\frac{p}{2})} + \frac{\frac{1}{6}\left(Z^2_{(k,m,\frac{p}{2})}-1\right)k_3(S)}{1+0.2k_3^2(S)}\quad , \quad Z_{(k,m,1-\frac{p}{2})} + \frac{\frac{1}{6}\left(Z^2_{(k,m,1-\frac{p}{2})}-1\right)k_3(S)}{1+0.2k_3^2(S)}$

Appendix B

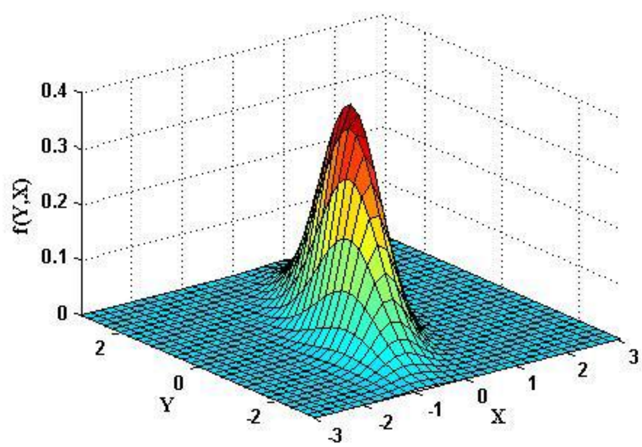
Central product moments of bivariate normal distribution

The pdf of bivariate normal distribution and probability density plot are

$$f(Y,X) = \frac{1}{2\pi\sigma_Y\sigma_X\sqrt{1-\rho_{YX}^2}} e^{\left\{\frac{-1}{2(1-\rho_{YX}^2)}\left[\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2+\left(\frac{X-\mu_X}{\sigma_X}\right)^2-2\rho_{YX}\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\left(\frac{X-\mu_X}{\sigma_X}\right)\right]\right\}},$$

$$-\infty < Y < \infty, -\infty < X < \infty, \sigma_Y > 0\ , \sigma_X > 0, -1 < \rho_{YX} < +1.$$

where, μ_Y and μ_X are population means of Y and X, respectively, σ_Y^2 and σ_X^2 are population variances of Y and X, respectively, ρ_{YX} denotes a population regression coefficient between Y and X and expressed as: $\rho_{YX} = \frac{\sigma_{YX}}{\sigma_Y\sigma_X}$, σ_{YX} represents the population covariance between Y and X.



Then central product moments of bivariate normal distribution $\eta_{pq(BN)} = E[Y - \mu_Y]^2[X - \mu_X]^2$ are given in the following table:

No	p	q	$\eta_{pq(BN)}$
1	2	0	σ_Y^2
2	0	2	σ_X^2
3	4	0	$3\sigma_Y^4$
4	0	4	$3\sigma_X^4$
5	6	0	$15\sigma_Y^6$
6	0	6	$3\sigma_Y^6$
7	1	1	$\rho_{YX}\sigma_Y\sigma_X$
8	2	2	$(1+2\rho_{YX}^2)\sigma_Y^2\sigma_X^2$
9	1	3	$3\rho_{YX}\sigma_Y\sigma_X^3$
10	3	1	$3\rho_{YX}\sigma_Y^3\sigma_X$
11	2	4	$3(1+4\rho_{YX}^2)\sigma_Y^2\sigma_X^4$
12	3	3	$3\rho_{YX}(3+\rho_{YX}^2)\sigma_Y^3\sigma_X^3$
13	4	2	$3(1+4\rho_{YX}^2)\sigma_Y^4\sigma_X^2$
14	1	5	$15\rho_{YX}\sigma_Y\sigma_X^5$
15	5	1	$15\rho_{YX}\sigma_Y^5\sigma_X$

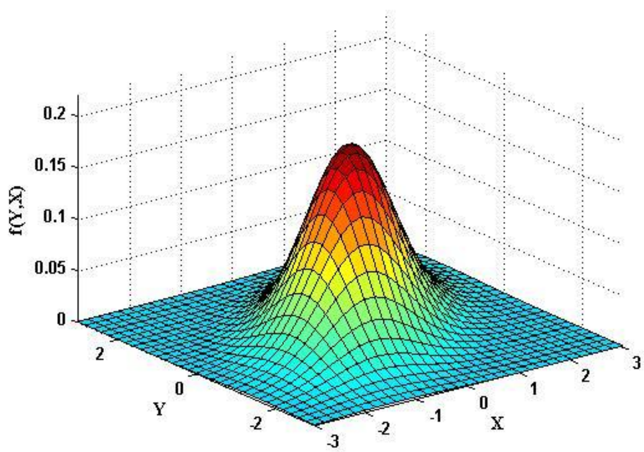
Central product moments of bivariate t distribution

The pdf of location scale bivariate t distribution and probability density plot are

$$f(Y,X) = \frac{1}{2\pi\sigma_Y\sigma_X\sqrt{1-\rho_{YX}^2}}\left[1+\frac{\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2+\left(\frac{X-\mu_X}{\sigma_X}\right)^2-2\rho\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\left(\frac{X-\mu_X}{\sigma_X}\right)}{v\left(1-\rho_{YX}^2\right)}\right]^{-\frac{(v+2)}{2}}$$

$$-\infty < Y < \infty, -\infty < X < \infty, \sigma_Y > 0\,, \sigma_X > 0, -1 < \rho_{YX} < +1.$$

where, μ_Y and μ_X are population means of Y and X, respectively, σ_Y^2 and σ_X^2 are population variances of Y and X, respectively, ρ_{YX} denotes a population regression coefficient between Y and X.



Then central product moments of bivariate t distribution $\eta_{pq(BT)} = E[Y-\mu_Y]^p[X-\mu_X]^q$ are given in table

No	p	q	$\eta_{pq(BT)}$
1	1	0	0
2	1	1	$\rho_{YX}\sigma_Y\sigma_X\gamma_2$
3	1	2	0
4	1	3	$3\sigma_Y\sigma_X^3\rho_{YX}\gamma_4$
5	5	1	0
6	1	5	$15\sigma_Y\sigma_X^6\rho_{YX}\gamma_6$
7	2	0	$\sigma_Y^2\gamma_2$
8	2	1	0
9	2	2	$(1+2\rho_{YX}^2)\sigma_Y^2\sigma_X^2\gamma_4$
10	2	3	0
11	2	4	$3\sigma_Y^2\sigma_X^4(1+4\rho_{YX}^2)\gamma_6$
12	3	0	0
13	3	1	$3\sigma_Y^3\sigma_X\rho_{YX}\gamma_4$
14	3	2	0
15	3	3	$3\sigma_Y^3\sigma_X^3\rho_{YX}(3+2\rho_{YX}^2)\gamma_6$
16	4	0	$3\sigma_Y^4\gamma_4$
17	4	1	0
18	4	2	$3\sigma_Y^4\sigma_X^2(1+4\rho_{YX}^2)\gamma_6$
19	5	0	0
20	5	1	$15\sigma_Y^5\sigma_X\rho_{YX}\gamma_6$
21	6	0	$15\sigma_Y^6\gamma_6$

where $\gamma_{2a} = \frac{\left(\frac{v}{2}\right)^{a}}{\left(\frac{v}{2}-1\right)^{\{a\}}}$, $v > 2a$, $\left(\frac{v}{2}-1\right)^{\{a\}} = \left[\left(\frac{v}{2}-1\right)\right]\left[\left(\frac{v}{2}-1\right)-1\right]\ldots\left[\left(\frac{v}{2}-1\right)-a+1\right]$.

Central product moments of bivariate lognormal distribution

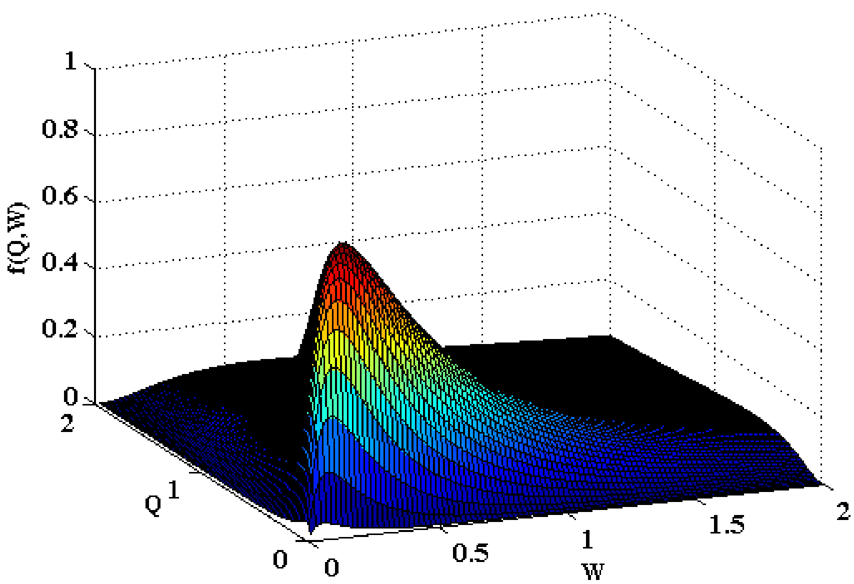
Two positive random variable Q and W with means and variances are μ_Q, μ_W σ_Q^2 and σ_W^2 respectively said to be bivariate lognormal distributed if $Y = lnQ$ and $X = lnW$ follows bivariate normally distributed with means and variances are μ_Y, μ_X σ_Y^2 and σ_X^2 .

The pdf of bivariate lognormal distribution and probability density plot are

$$f(Q,W) = \frac{1}{2\pi\sigma_Y\sigma_X\sqrt{1-\rho_{YX}^2}} e^{\left\{\frac{-1}{2(1-\rho_{YX}^2)}\left[\left(\frac{\log(Q)-\mu_Y}{\sigma_Y}\right)^2+\left(\frac{\log(W)-\mu_X}{\sigma_X}\right)^2-2\rho\left(\frac{\log(Q)-\mu_Y}{\sigma_Y}\right)\left(\frac{\log(W)-\mu_X}{\sigma_X}\right)\right]\right\}},$$

$$0 < Q < \infty, 0 < W < \infty, \sigma_Y > 0\,, \sigma_X > 0, -1 < \rho_{YX} < +1.$$

where $\mu_Y = l\,n(\mu_Q) - \left(\frac{\sigma_Q^2}{2}\right)$, $\mu_X = l\,n(\mu_W) - \left(\frac{\sigma_W^2}{2}\right)$, $\sigma_Y^2 = ln\left(1+\frac{\sigma_Q^2}{\mu_Q^2}\right)$ and $\sigma_X^2 = ln\left(1+\frac{\sigma_W^2}{\mu_W^2}\right)$.



Then central product moments of bivariate lognormal distribution $\eta_{pq(BLN)} = E[Y-\mu_Y]^p[X-\mu_X]^q$ are given in following table:

No	p	q	$\eta_{pq(BLN)}$
1	1	0	0
2	1	1	$exp\left[\mu_Y+\mu_X+\frac{1}{2}(\sigma_Y^2+\sigma_X^2+2\rho_{YX}\sigma_Y\sigma_X)\right]$- $exp\left[\mu_Y+\mu_X+\frac{1}{2}(\sigma_Y^2+\sigma_X^2)\right]$
3	2	2	$exp[2\mu_Y+2\mu_X+\sigma_Y^2+\sigma_X^2][[exp[\sigma_Y^2+\sigma_X^2+2\rho_{YX}\sigma_Y\sigma_X] + exp[\sigma_Y^2+\sigma_X^2]-2[exp[\sigma_X^2+2\rho_{YX}\sigma_Y\sigma_X] + exp[\sigma_Y^2]-2\exp[\sigma_X^2+2\rho_{YX}\sigma_Y\sigma_X]-3 - 2\exp[\sigma_Y^2+2\rho_{YX}\sigma_Y\sigma_X]+4\exp[\rho_{YX}\sigma_Y\sigma_X]]$
4	0	1	0
5	4	0	$(\exp[\sigma_Y^2]+2)\sqrt{(\exp[\sigma_Y^2]-1)}$
6	0	4	$(\exp[\sigma_X^2]+2)\sqrt{(\exp[\sigma_X^2]-1)}$
7	2	0	$exp[\sigma_Y^2-1]\exp[2\mu_Y+\sigma_X^2]$
8	0	2	$exp[\sigma_X^2-1]\exp[2\mu_X+\sigma_Y^2]$

Appendix C

Theorem 1: when population of interests follows bivariate normal distribution, then population regression coefficient between sample variances $b_{s^2_{Y(SRS,BN),j}s^2_{X(SRS,BN),j}}$ is equal to the square of the population regression coefficient between Y and X i.e. $b_{s^2_{Y(SRS,BN),j}s^2_{X(SRS,BN),j}} = \frac{\sigma_Y^2}{\sigma_X^2}\rho_{YX}^2 = (b_{YX})^2$.

Proof: Let $(Y_1,X_1),(Y_2,X_2),(Y_3,X_3),\dots,(Y_n,X_n)$ denote a sample of paired observations of size *n* follows bivariate normal distribution with population means and variances are $\mu_{Y(BN)},\mu_{X(Bn)}$, $\sigma^2_{Y(BN)}$ and $\sigma^2_{X(BN)}$. Then, population regression coefficient between $s^2_{Y(SRS,BN),j}$ and $s^2_{X(SRS,BN),j}$ is defined as (see Riaz et al., 2014):

$$b_{s^2_{Y(SRS,BN),j}s^2_{X(SRS,BN),j}} = \frac{\sigma^2_{Y(BN)}}{\sigma^2_{X(BN)}}\left[\frac{\mu_{22(BN)}-1}{\mu_{04(BN)}-1}\right], \quad \text{(C1)}$$

where $$\mu_{22(BN)} = \frac{\eta_{22(BN)}}{\sqrt{(\eta_{20(BN)})^2(\eta_{02(BN)})^2}},$$

$\eta_{22(BN)} = E(Y-\mu_{Y(BN)})^2(X-\mu_{X(BN)})^2$,$\eta_{20(BN)} = E(Y-\mu_{Y(BN)})^2$ and $\eta_{02(BN)} = E(X-\mu_{X(BN)})^2$ and $\mu_{04(BN)}$ is kurtosis of X respectively.

From (C1) $\eta_{22(BN)}$, $\eta_{20(BN)}$, $\eta_{02(BN)}$ are the product central moments of bivariate normal distribution (see, Appendix B). Further we proceed (C1) in order to get the final result as follows:

$$\eta_{22(BN)} = E(Y-\mu_{Y(BN)})^2(X-\mu_{X(BN)})^2,$$

$$\eta_{22(BN)} = \sigma^2_{Y(BN)}\sigma^2_{X(BN)}(2\rho^2_{YX}+1).$$

Kurtosis of X $\mu_{04} = \frac{\mu_4}{\mu_2^2} = 3.$

Now $\mu_{22(BN)}$ becomes,

$$\mu_{22(BN)} = \frac{\sigma^2_{Y(BN)}\sigma^2_{X(BN)}(2\rho^2_{YX}+1)}{\sqrt{(\sigma^2_{Y(BN)})^2(\sigma^2_X(BN))^2}},$$

$$\mu_{22(BN)} = 2\rho^2_{YX}+1.$$

Finally (B1) becomes, $b_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} = \frac{\sigma^2_{Y(BN)}[2\rho^2_{YX}+1-1]}{\sigma^2_{X(BN)}[3-1]}$,

$b_{s^2_{Y(SRS,BN),j}s^2_{X(SRS,BN),j}} = \left(\frac{\sigma_{Y(BN)}}{\sigma_{X(BN)}}\rho_{YX}\right)^2$. This shows that population regression coefficient between $s^2_{Y(SRS,BN),j}$ and $s^2_{X(SRS,BN),j}$ is equal to the square of the population regression coefficient between $Y$and X.

Corollary 1: Population correlation coefficient between $s^2_{Y(SRS,BN),j}$ and $s^2_{X(SRS,BN),j}$ is equal to the square of the population correlation coefficient between Y and X i.e. $\rho_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} = \rho^2_{YX}$.

Proof: By definition population regression coefficient between sample variances $b_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}}$ is defined as:

$$b_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} = \rho_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} \frac{\sigma_{s^2_{Y(SRS,BN),j}}}{\sigma_{s^2_{X(SRS,BN),j}}}, \tag{C2}$$

where $\sigma_{s^2_{Y(SRS,BN),j}} = \sqrt{\frac{2}{n-1}\sigma^4_{Y(BN)}}$ and $\sigma_{s^2_{X(SRS,BN),j}} = \sqrt{\frac{2}{n-1}\sigma^4_{X(BN)}}$.

We have proved that $b_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} = \frac{\sigma^2_{Y(BN)}}{\sigma^2_{X(BN)}}\rho^2_{YX}$ using theorem 1 (C3)

So, From C2 and C3,

$$\rho_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} \frac{\sigma_{s^2_{Y(SRS,BN),j}}}{\sigma_{s^2_{X(SRS,BN),j}}} = \frac{\sigma^2_{Y(BN)}}{\sigma^2_{X(BN)}} \rho^2_{YX},$$

$$\rho_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} \sqrt{\frac{\frac{2}{n-1}\sigma^4_{Y(BN)}}{\frac{2}{n-1}\sigma^4_{X(BN)}}} = \frac{\sigma^2_{Y(BN)}}{\sigma^2_{X(BN)}} \rho^2_{YX}\ ,$$

$\rho_{s^2_{Y(SRS,BN),j}\, s^2_{X(SRS,BN),j}} = \rho^2_{YX}$. This shows that population correlation coefficient between $s^2_{Y(SRS,BN),j}$ and $s^2_{X(SRS,BN),j}$is equal to the square of the population correlation coefficient between Y and X.

Corollary 2: $Cov\left(s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}\right) = \frac{2}{n-1}[Cov(Y,X)]^2$.

Proof: By definition $Cov\left(s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}\right)$ is defined as:

$$\rho_{s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}} = \frac{Cov\left(s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}\right)}{\sqrt{Var(s^2_{Y(SRS,BN),j})Var(s^2_{X(SRS,BN),j})}}.$$

Rewriting the above equation and substituting the values of $Var(s^2_{Y(SRS,BN),j})$ and $Var(s^2_{X(SRS,BN),j}$, we obtain

$$Cov\left(s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}\right) = \frac{2}{n-1} \rho_{s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}} \sigma^2_{Y(BN)} \sigma^2_{X(BN)}.$$

From corollary 1 we substitute the value of $\rho_{s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}}$ in the above equation and obtain

$Cov\left(s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}\right) = \frac{2}{n-1} \rho^2_{YX} \sigma^2_{Y(BN)} \sigma^2_{X(BN)}$, and finally, we get our result

$$Cov\left(s^2_{Y(SRS,BN),j}\, , s^2_{X(SRS,BN),j}\right) = \frac{2}{n-1}[Cov(Y,X)]^2.$$

Theorem 2: When population of interest is bivariate t then population regression coefficient between sample variances $(s^2_{Y(SRS,BT),j}$ and $, s^2_{X(SRS,BT),j})$ is equal to $\frac{\sigma^2_{Y(BT)}}{\sigma^2_{X(BT)}} \left[\rho^2_{YX}\left(1 - \frac{1}{v-1}\right) + \frac{1}{v-1}\right]$.

Let $(Y_1, X_1), (Y_2, X_2), (Y_3, X_3), \ldots, (Y_n, X_n)$ denote a sample of paired observations of size n follows bivariate t distribution with population means and variances are $\mu_{Y(BT)}, \mu_{X(BT)}$,

$\sigma^2_{Y(BT)}$ and $\sigma^2_{X(BT)}$Then, population regression coefficient $b_{s^2_{Y(SRS,BT),j}\, s^2_{X(SRS,BT),j}}$ between $s^2_{Y(SRS,BT),j}$and $s^2_{X(SRS,BT),j}$ is defined as (see Riaz et al., 2014):

$$b_{s^2_{Y(SRS,BT),j} s^2_{X(SRS,BT),j}} = \frac{\sigma^2_{Y(BT)}}{\sigma^2_{X(BT)}}\left[\frac{\mu_{22(BT)}-1}{\mu_{04(BT)}-1}\right], \qquad \text{(C4)}$$

where $\mu_{22(BT)} = \frac{\eta_{22(BT)}}{\sqrt{(\eta_{20(BT)})^2(\eta_{02(BT)})^2}}$, $\eta_{22(BT)} = E(Y-\mu_{Y(BT)})^2(X-\mu_{X(BT)})^2$,$\eta_{20(BT)} = E(Y-\mu_{Y(BT)})^2$ and $\eta_{02(BT)} = E(X-\mu_{X(BT)})^2$ and $\mu_{04(BT)}$ is kurtosis of X respectively.

From (C4) $\eta_{22(BT)}$, $\eta_{20(BT)}$, $\eta_{02(BT)}$ are the product central moments of bivariate t distribution (see, Appendix B). Further we proceed (C4) in order to get the final result as follows:

$$\eta_{22(BT)} = E(Y-\mu_{Y(BT)})^2(X-\mu_{X(BT)})^2,$$

$\eta_{22(BT)} = \frac{\sigma^2_{Y(BN)}\sigma^2_{X(BN)}(2\rho^2_{YX}+1)\gamma_4}{\gamma_2^2}$, where

$\gamma_4 = \frac{v^2}{(v-2)(v-4)}$, $\gamma_2 = \frac{v}{v-2}$ and v is degree of freedom.

Kurtosis of X $\mu_{04(BT)} = \frac{\mu_4}{\mu_2^2} = 3\frac{\gamma_4}{\gamma_2^2}$.

Now $\mu_{22(BT)}$ becomes,

$$\mu_{22(BN)} = \frac{\frac{\sigma^2_{Y(BN)}\sigma^2_{X(BN)}(2\rho^2_{YX}+1)\gamma_4}{\gamma_2^2}}{\sqrt{(\sigma^2_{Y(BN)}\,\gamma_2\,)^2(\sigma^2_{X(BN)}\gamma_2\,))^2}},$$

$$\mu_{22(BN)} = (2\rho^2_{YX}+1)\frac{\gamma_4}{\gamma_2^2}.$$

Substituting $\mu_{22(BT)}$ and $\mu_{04(BT)}$ in (C4)

$$b_{s^2_{Y(SRS,BT),j}\, s^2_{X(SRS,BT),j}} = \frac{\sigma^2_{Y(BT)}}{\sigma^2_{X(BT)}}\left[\frac{(2\rho^2_{YX}+1)\frac{\gamma_4}{\gamma_2^2}-1}{3\frac{\gamma_4}{\gamma_2^2}-1}\right],$$

After simplification we get,

$$b_{s^2_{Y(SRS,BT),j}\, s^2_{X(SRS,BT),j}} = \frac{\sigma^2_{Y(BT)}}{\sigma^2_{X(BT)}}\left[\rho^2_{YX}\left(1-\frac{1}{v-1}\right)+\frac{1}{v-1}\right]$$

Nomenclature

ρ and ρ_{YX}	Population correlation coefficients between study variable Y and auxiliary variable X.
μ_X	Population mean of X
μ_Y	Population mean of Y
σ_X	Population standard deviation of X
σ_Y	Population standard deviation of Y
$k-m$	Decision observations used in a given rule
k	Total Consecutive points to be considered
p	Probability of single point falling outside the respective limits
α	False alarm rate
δ	Shift in the process parameter
d_2	Mean of the sampling distribution of relative range $\frac{R}{\sigma}$
d_3	Standard deviation of the sampling distribution of relative range $\frac{R}{\sigma}$
e_2	Mean of the sampling distribution of relative range $\frac{S}{\sigma}$
e_3	Standard deviation of the sampling distribution of relative range $\frac{S}{\sigma}$
c_4^*	Skewness adjustment factor for location charts
d_4^*	Skewness adjustment factor for R chart
e_4^*	Skewness adjustment factor for S chart
k_3	Skewness of the study variable
k_3	Ratio of the standard deviation of the statistic and standard deviation of study variable Y
$Z_{(\frac{\alpha}{2})}$	$\left(\frac{\alpha}{2}\right)$ th quantile of standard normal distribution
$Z_{(1-\frac{\alpha}{2})}$	$\left(1-\frac{\alpha}{2}\right)$ th quantile of standard normal distribution
σ_R	Standard deviation of the sampling distribution of R
σ_S	Standard deviation of the sampling distribution of S

References

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